Chapter 6

FUNCTIONS ON THE CIRCLE (FOURIER ANALYSIS)

In this chapter we shall study **periodic functions** of a real variable. The importance of such functions derives from the fact that many natural and physical phenomena are oscillatory, or recurrent. In the early 19th century, J. B. J. Fourier laid down the foundations of the study of periodic functions in his treatise *Analytic Theory of Heat.* There remained a few gaps and difficulties in Fourier's theory and much mathematical energy during the 19th century was expended in the study of these problems. The invention of Lebesgue's theory of integration in the early 20th century finally laid the foundations to this theory. Our exposition will not follow this chronological pattern; but rather will try to develop the way of thinking about Fourier series which emerged during the late 19th century.

A periodic function is one whose behavior is recurrent. That is, there is a certain number L, called the **period of the function**, such that the function repeats itself over every interval of length L,

$$f(x + L) = f(x) \qquad \text{for all } x \in R$$

From our point of view (which is very much *a posteriori*) the study of periodic functions begins by discarding the notion of periodicity in favor of a change in the *geometry* of the domain. That is, to study the collection of all periodic functions with a fixed period, we make the underlying space periodic instead. We shall think of the real line as wound around a circle, and our periodic functions are just the functions on the circle.

To fix the ideas, we shall have a particular circle in mind: the set Γ of complex numbers of modulus one. We have already seen that there is a mapping $\theta \to \cos\theta + i\sin\theta = e^{i\theta}$ of the real numbers onto Γ which is one-to-one on an interval of length 2π, except that both end points go onto the same point. This mapping does precisely what we want: It winds the real line around Γ. A continuous function on Γ is a function of $e^{i\theta}$ which varies continuously with θ. Thus the continuous functions on Γ are precisely the continuous functions on R which are periodic of period 2π:

$$f(x + 2\pi) = f(x) \qquad \text{for all } x \in R$$

In the past few chapters we have been studying the behavior of functions from the point of view of differentiation. We have studied the Taylor expansion, an expansion into polynomials, and we have related the coefficients to the subsequent derivatives of the function. Since the simplest periodic functions are the trigonometric polynomials, we attempt to expand a given periodic function in a series of trigonometric polynomials. This is the so-called **Fourier series** of the function. The interesting fact here is that the relevant coefficients are found by integration. In fact, as we shall see, the Fourier series of a function is a sort of an expansion in terms of an orthonormal basis in the vector space of continuous functions on the circle with the inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)g(\theta)\,d\theta$$

Finally, as the circle is the set of complex numbers of modulus one, it is the boundary of the unit disk in C and we can study the relation between Taylor expansions in the disk and the Fourier expansions on the circle for suitable functions. It will turn out that for such functions the Taylor coefficients can also be obtained by integration on the circle.

6.1 Approximation by Trigonometric Polynomials

We shall begin with the attitude that we are studying complex-valued functions on the circle. According to this view, the function $e^{i\theta}$ is the simplest and the most basic function. This attitude is really just a convenience; the point of view of strictly real-valued functions would consign us to consider $\cos\theta$, $\sin\theta$ as the elementary building blocks of our theory.

But, since $e^{i\theta} = \cos\theta + i\sin\theta$, there is little difference, and we select the more comfortable notation.

Our purpose is to describe a given function on the circle in terms of the powers of $e^{i\theta}$, both positive and negative. More precisely, if the series

$$\sum_{n=-\infty}^{\infty} a_n e^{in\theta} \tag{6.1}$$

converges for all θ, it defines a function on the circle. We ask the converse question: Can we express any periodic function as such a series? If only finitely many of the $\{a_n\}$ in (6.1) are nonzero, there is no problem of convergence, and the sum defines a function, called a **trigonometric polynomial.** This subject gets off the ground once we know how to compute the $\{a_n\}$ from the given function, and that leads us to our first proposition.

Proposition 1. *Let $P(\theta) = \sum_{n=-N}^{N} a_n e^{in\theta}$ be a trigonometric polynomial. Then*

$$a_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(\theta)e^{-im\theta}\,d\theta$$

for all m.

Proof.

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} P(\theta)e^{-im\theta}\,d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-N}^{N} a_n e^{in\theta}\right)e^{-im\theta}\,d\theta \\ &= \frac{1}{2\pi}\sum_{n=-N}^{N} a_n \int_{-\pi}^{\pi} e^{i(n-m)\theta}\,d\theta \\ &= \frac{1}{2\pi}a_m\cdot 2\pi + 0 = a_m\end{aligned}$$

Now, given a continuous function on the circle, if it has an expansion into a series of trigonometric polynomials, we could expect that the coefficients of this series will be related to the function in the same way. Thus we form this definition.

Definition 1. Let f be a continuous function on the circle. The nth **Fourier coefficient** of f is

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)e^{-in\phi}\,d\phi \tag{6.2}$$

The **Fourier series** of f is the series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta} \tag{6.3}$$

Examples

1. Let $f(\theta) = \sin\theta$. Since

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

its Fourier series is

$$\frac{-1}{2i}e^{-i\theta} + \frac{1}{2i}e^{i\theta}$$

From (6.2) we can deduce (as is also easily computed):

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\theta\, e^{i\theta}\, d\theta = \frac{-1}{2i} \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\theta\, e^{-i\theta}\, d\theta = \frac{1}{2i}$$

2. Since $\cos m\theta = \frac{1}{2}(e^{im\theta} + e^{-im\theta})$, the Fourier series of $\cos m\theta$ is

$$\tfrac{1}{2}e^{-im\theta} + \tfrac{1}{2}e^{im\theta}$$

3. Let $f(\theta) = \cos^2\theta$. Then

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2\phi e^{-in\phi}\, d\phi = \frac{1}{4\pi}\int_{-\pi}^{\pi}(1 + \cos 2\phi)e^{-in\phi}\, d\phi$$

$$(n) = \begin{cases} \frac{1}{4} & n = -2, 2 \\ \frac{1}{2} & n = 0 \\ 0 & n \neq -2, 0, 2 \end{cases}$$

Thus the Fourier series of $\cos^2\theta$ is

$$\tfrac{1}{4}e^{-i2\theta} + \tfrac{1}{2} + \tfrac{1}{4}e^{i2\theta}$$

(Notice that $\cos^2\theta = 1/2(1 + \cos 2\theta) = 1/2(1 + 1/2(e^{i\theta} + e^{-i\theta}))$ is a trigonometric polynomial.)

4. Let $f(\theta) = \pi^2 - \theta^2$.

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi^2 - \phi^2)e^{-in\phi}\, d\phi$$

$$\hat{f}(0) = \frac{\pi^2}{2\pi}\int_{-\pi}^{\pi} d\phi - \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi^2\, d\phi = \frac{2\pi^2}{3} \qquad n = 0$$

$$\hat{f}(n) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \phi^2 e^{-in\phi}\, d\phi = (-1)^n \frac{2}{n^2} \qquad n \neq 0$$

by two integrations by parts. Thus the Fourier series of $\pi^2 - \theta^2$ is

$$\frac{2\pi^2}{3} + 2\sum_{n\neq 0} \frac{(-1)^n}{n^2} e^{in\theta} \tag{6.4}$$

Notice that by the comparison test, this series does converge to a continuous function of $e^{i\theta}$:

$$\frac{2\pi^2}{3} + 2\sum_{n\neq 0} (-1)^n \frac{(e^{i\theta})^n}{n^2}$$

In order to conclude that this is the given function $\pi^2 - \theta^2$, we shall need more theoretical investigations.

5. It is not necessary for a function to be continuous to have a Fourier expansion. It need only be integrable for the expressions (6.2), (6.3) to be computable. Let us compute the Fourier series of

$$f(\theta) = \begin{cases} 1 & \theta \geq 0 \\ 0 & \theta < 0 \end{cases}$$

$$\hat{f}(0) = \frac{1}{2\pi}\int_0^{\pi} d\phi = \frac{1}{2}$$

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{\pi} e^{in\phi}\, d\phi = \frac{-1}{2\pi i n} e^{-in\phi}\Big|_0^{\pi} = \frac{-1}{2\pi i n}[e^{-in\pi} - 1]$$

$$= \begin{cases} 0 & n \text{ even, } n \neq 0 \\ \dfrac{1}{\pi i n} & n \text{ odd} \end{cases}$$

Thus the Fourier series of f is

$$\frac{1}{2}+\frac{1}{\pi i}\sum_{\substack{n=-\infty\\ n\text{ odd}}}^{\infty}\frac{e^{in\theta}}{n} \tag{6.5}$$

Recapturing the Function from Its Fourier Series

Notice that no claim of convergence in Definition 1 is made. In particular, the series (6.5) appears not to converge, for the comparison test does not apply. However, we cannot conclude that convergence fails; only that the question can be exceedingly difficult. We ask instead what appears to be a simpler question: Does the Fourier series identify the given function, and if so, in what way? We now try to investigate the recapture of a function by its Fourier series, deliberately leaving aside all questions of convergence.

Let f be a given integrable function on the circle and consider the "function"

$$g(\theta)=\sum_{n=-\infty}^{\infty}\hat{f}(n)e^{in\theta}$$

By definition of $\hat{f}(n)$,

$$g(\theta)=\sum_{n=-\infty}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\phi)e^{in(\theta-\phi)}\,d\phi$$

Now we interchange $\sum$ and $\int$, obtaining

$$g(\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\phi)\sum_{n=-\infty}^{\infty}e^{in(\theta-\phi)}\,d\phi$$

Well, it is too bad it turned out this way because we are still up against a convergence problem, like it or not. In fact, the situation is worse: it is untenable because

$$\sum_{n=-\infty}^{\infty}e^{in(\theta-\phi)} \tag{6.6}$$

converges for *no* values of ϕ. This seemingly insurmountable obstacle can be overcome, so long as we are not solely interested in pointwise convergence, by a subtle mathematical technique: that of inserting convergence factors.

If we replace the series (6.6) by the series

$$\sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-\phi)} \tag{6.7}$$

this series converges beautifully for $r < 1$ and the series (6.6) is in some ideal sense the limit of (6.7) as r tends to 1. Stepping backward two steps, this causes us to now consider the series

$$g(r, \theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta} \tag{6.8}$$

and the limit $\lim_{r \to 1} g(r, \theta)$ (hoping of course that it is $f(\theta)$). Notice that the series (6.8) does converge since the Fourier coefficients $\{\hat{f}(n)\}$ are bounded (Problem 1) and the comparison test applies. Now, proceeding as above but this time with $g(r, \theta)$, we obtain

$$g(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-\phi)}\, d\phi \qquad r < 1$$

and here we *can* interchange $\sum$ and $\int$ because the series in question converges uniformly. The sum in the above integral can be put in a nicer form since it is a sum of two geometric series.

$$\begin{aligned} P(r, t) &= \sum_{n=-\infty}^{\infty} r^{|n|} e^{int} = \sum_{n=1}^{\infty} (re^{-it})^n + \sum_{n=0}^{\infty} (re^{it})^n \\ &= \frac{1}{1 - re^{-it}} - 1 + \frac{1}{1 - re^{it}} \qquad (6.9) \\ &= \frac{1 - r^2}{1 + r^2 - r(e^{it} + e^{-it})} = \frac{1 - r^2}{1 + r^2 - 2r \cos t} \end{aligned}$$

The function $P(r, t)$ is called **Poisson's kernel** (named after its French discoverer, not because its whole technique is fishy), and the association of f to g is called the **Poisson transform.** Thus, the Poisson transform

$$(Pf)(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - \phi)}\, d\phi = \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta} \tag{6.10}$$

takes continuous functions on the circle into continuous functions of r, θ for $|r| < 1$; that is, into continuous functions on the open unit disk. We shall later see the importance of the Poisson transform from the point of view of partial differential equations.

Examples (Some Poisson Transforms)

6. We can find the Poisson transform of functions on the circle quite explicitly, using some complex notation and Equation (6.10). For example, consider $f(\theta) = \cos^2 \theta$. Using Example 3 we have

$$Pf(r, \theta) = \tfrac{1}{4}r^2 e^{-i2\theta} + \tfrac{1}{2} + \tfrac{1}{4}r^2 e^{i2\theta} = \tfrac{1}{2}[1 + \tfrac{1}{2}(re^{-i\theta})^2 + \tfrac{1}{2}(re^{i\theta})^2]$$

Thinking of r, θ as polar coordinates in the disk, we can rewrite this (using $z = re^{i\theta} = x + iy$, $\bar{z} = re^{-i\theta} = x - iy$):

$$Pf(z) = \tfrac{1}{2}[1 + \tfrac{1}{2}(z^2 + \bar{z}^2)] = \tfrac{1}{2}[1 + \operatorname{Re} z^2] = \tfrac{1}{2}(1 + x^2 - y^2)$$

Clearly,

$$\lim_{r \to 1} Pf(r, \theta) = \lim_{x^2 + y^2 \to 1} Pf(z) = \frac{1}{2}(1 + x^2 - (1 - x^2)) = x^2 = \cos^2 \theta$$

7. The Poisson transform of

$$f(\theta) = \begin{cases} 1 & \theta \geq 0 \\ 0 & \theta < 0 \end{cases}$$

is given by

$$Pf(r, \theta) = \frac{1}{2} + \frac{1}{\pi i} \sum_{n \text{ odd}} \frac{r^{|n|} e^{in\theta}}{n} = \frac{1}{2} + \frac{2}{\pi} \sum_{\substack{n \text{ odd} \\ n > 0}} \frac{1}{2i} \left(\frac{r^n e^{in\theta}}{n} - \frac{r^n e^{-in\theta}}{n} \right)$$

$$Pf(z) = \frac{1}{2} + \frac{2}{\pi} \operatorname{Im} \left(\sum_{\substack{n \text{ odd} \\ n > 0}} \frac{z^n}{n} \right) = \frac{1}{2} + \frac{2}{\pi} \operatorname{Im} \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}$$

Now, we can use Taylor expansions to obtain a closed form for this series.

$$\sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1} = \int \sum_{n=0}^{\infty} z^{2n}\, dz = \int \frac{dz}{1 - z^2}$$

Now

$$\int \frac{dz}{1-z^2} = \frac{1}{2}\left(\int \frac{dz}{1-z} + \int \frac{dz}{1+z}\right) = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

(We have used real-variable techniques to find this closed form, but once it is found it is valid for all z, $|z| \le 1$.) Thus

$$Pf(z) = \frac{1}{2} + \frac{1}{\pi}\operatorname{Im}\ln\left(\frac{1+z}{1-z}\right)$$

As $|z| \to 1$, $Pf(z)$ has a limit except for $z \to 1$, $z \to -1$. We shall now show that except for these two values, $\lim_{r\to 1} Pf(r, \theta) = f(\theta)$.

$$\lim_{r\to 1} Pf(r, \theta) = Pf(1, \theta) = \frac{1}{2} + \frac{1}{\pi}\operatorname{Im}\ln\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right) \tag{6.11}$$

Now

$$\frac{1+e^{i\theta}}{1-e^{i\theta}} = \frac{(1+e^{i\theta})(1-e^{-i\theta})}{(1-e^{i\theta})(1-e^{-i\theta})} = \frac{1+e^{i\theta}-e^{-i\theta}-1}{1-e^{i\theta}-e^{-i\theta}+1}$$

$$= \frac{i\sin\theta}{(1-\cos\theta)} \tag{6.12}$$

Since $\ln z = \ln|z| + i\arg z$, $\operatorname{Im}\ln z = \arg z$ for any complex number. Since (6.12) is pure imaginary, we have

$$\operatorname{Im}\ln\frac{1+e^{i\theta}}{1-e^{i\theta}} = \begin{cases} \dfrac{\pi}{2} & \theta > 0 \\[2ex] -\dfrac{\pi}{2} & \theta < 0 \end{cases}$$

Thus, referring back to (6.11)

$$\lim_{r\to 1} Pf(r, \theta) = \frac{1}{2} + \frac{1}{\pi}\left(\frac{\pi}{2}\right) = 1 \qquad \text{if } \theta > 0$$

$$= \frac{1}{2} + \frac{1}{\pi}\left(-\frac{\pi}{2}\right) = 0 \qquad \text{if } \theta < 0$$

We are still hoping that it is true for all f that $\lim_{r \to 1} Pf(r, \theta) = f(\theta)$. Of course, this turns out to be true. To see this we have to verify some properties of Poisson's kernel. First we rewrite the Poisson kernel as

$$P(r, t) = \frac{1 - r^2}{(1 - r)^2 + 2r(1 - \cos t)}$$

From this reformulation we easily conclude the following properties:

(i) $P(r, t) \geq 0 \qquad$ for all values of $r, t, r < 1$

(ii) $P(r, 0) = \dfrac{1 - r^2}{(1 - r)^2} = \dfrac{1 + r}{1 - r} \to \infty$ as $r \to 1$

(iii) On the other hand, for values of $t \neq 0$, $P(r, t) \to 0$ as $r \to 1$. If $|t| \geq \delta$,

$$P(r, t) = \frac{1 - r^2}{(1 - r)^2 + 2r(1 - \cos \delta)} \leq \frac{1 - r^2}{2r(1 - \cos \delta)} \to 0$$

uniformly as $r \to 1$.

For a fixed value of r, the graph of $P(r, t)$ is drawn in Figure 6.1. As $r \to 1$, the peak goes up and the valleys get larger and deeper. Finally,

(iv) $\dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} P(r, t)\, dt = 1$

This can be computed directly; however it is easier to use Equation (6.10) in the particular case where f is the function which is identically one (see Problem 2).

Theorem 6.1. *If f is a continuous function on the circle,*

$$\lim_{r \to 1} Pf(r, \theta) = f(\theta)$$

Proof. Using property (iv) above we can write $Pf(r, \theta) - f(\theta)$ as an integral,

$$(Pf)(r, \theta) - f(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(\phi) - f(\theta)] P(r, \theta - \phi)\, d\phi$$

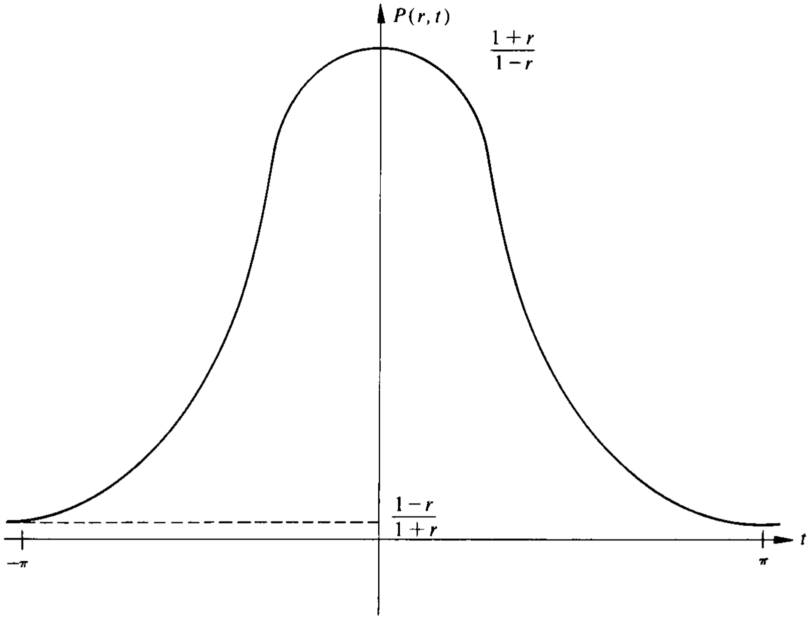

Figure 6.1

For any $\delta > 0$ we break up the integral into two pieces:

$$(Pf)(r, \theta) - f(\theta) = \frac{1}{2\pi}\int_{|\phi-\theta|\leq\delta} [f(\phi) - f(\theta)]P(r, \theta - \phi)\, d\phi$$

$$+ \frac{1}{2\pi}\int_{|\phi-\theta|\geq\delta} [f(\phi) - f(\theta)]P(r, \theta - \phi)\, d\phi$$

Now, by (iii) the integrand in the lower integral tends to zero as $r \to 1$ and, by continuity, $|f(\phi) - f(\theta)|$ is small for all ϕ near enough to θ so that we can make the first integral small by taking δ small.

More precisely, let $\varepsilon > 0$ be given. Let δ be such that

$$|f(\phi) - f(\theta)| < \frac{\varepsilon}{2} \qquad \text{if } |\phi - \theta| < \delta \tag{6.13}$$

Given that δ, by (iii), there is an $\eta > 0$ such that for $|r - 1| < \eta$,

$$\int_{|\phi-\theta|\geq\delta} P(r, \phi - \theta)\, d\phi < \frac{\pi\varepsilon}{2\,\|f\|_\infty}$$

Then for $|r-1|<\eta$,

$$|Pf(r,\theta)-f(r,\theta)| \le \frac{1}{2\pi}\int_{|\phi-\theta|\le\delta}|f(\phi)-f(\theta)|\,P(r,\theta-\phi)\,d\phi$$

$$+\frac{1}{2\pi}\int_{|\phi-\theta|\ge\delta}|f(\phi)-f(\theta)|\,P(r,\theta-\phi)\,d\phi$$

$$\le\frac{1}{2\pi}\cdot\frac{\varepsilon}{2}\cdot\int_{|\phi-\theta|\le\delta}P(r,\theta-\phi)\,d\phi$$

$$+\frac{1}{2\pi}\,2\,\|f\|_\infty\int_{|\phi-\theta|\ge\delta}P(r,\theta-\phi)\,d\phi$$

$$\le\frac{\varepsilon}{2}\cdot 1+\frac{\|f\|_\infty}{\pi}\,\frac{\pi\varepsilon}{2\,\|f\|_\infty}=\varepsilon$$

We seem to have come a long way away from our original quest, but we have not really. The content of Theorem 6.1 is this: Let f be a continuous function on the circle. Its Fourier series

$$\sum_{n=-\infty}^{\infty}\hat{f}(n)e^{in\theta}$$

is too hard to study as regards convergence, but it does represent f in some relevant sense. It "almost converges" to f; that is, if we put in factors to ensure the convergence and consider instead

$$Pf(r,\theta)=\sum_{n=-\infty}^{\infty}\hat{f}(n)r^{|n|}e^{in\theta}$$

then for r very close to 1, this function is very close to f. This allows us to make important assertions based on any information on the Fourier series of f. For example,

Collorary 1. *If f is a continuous function on the circle, and $\sum_{n=-\infty}^{\infty}|\hat{f}(n)|<\infty$, then f is the sum of its Fourier series,*

$$f(\theta)=\sum_{n=-\infty}^{\infty}\hat{f}(n)e^{in\theta}$$

Proof. The condition allows us to conclude on the basis of the comparison test that the Fourier series converges; the essential content here is that it converges to f.

In fact, by the comparison test, we can conclude that

$$(Pf)(r, \theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta}$$

is a continuous function on the closed unit disk: all $r \le 1$. Then for any θ, by Theorem 6.1.

$$f(\theta) = \lim_{r \to 1} Pf(r, \theta) = \lim_{r \to 1} \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta} = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in\theta}$$

In particular, if $\hat{f}(n)$ vanishes for all but finitely many n, then f is a trigonometric polynomial. Thus the trigonometric polynomials are precisely the class of continuous functions on the circle with only finitely many nonzero Fourier coefficients. A more basic consequence is that a function is uniquely determined by its Fourier series.

Collorary 2. *If f and g are continuous on the circle and $\hat{f}(n) = \hat{g}(n)$ for all n, then $f = g$.*

Proof. $f - g$ is continuous on Γ, and $(f-g)\hat{}(n) = \hat{f}(n) - \hat{g}(n) = 0$ for all n. Applying the first corollary to $f - g$ we see that it is the sum of its Fourier series, which is identically zero. Thus $f - g = 0$, so $f = g$.

Conditions on the Fourier coefficients of a function, such as that in Corollary 1, are not hard to come by. For example, suppose f is a twice continuously differentiable periodic function. Then by integrating by parts we have

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) e^{-in\phi}\, d\phi = \frac{1}{2\pi i n} \int_{-\pi}^{\pi} f'(\phi) e^{-in\phi}\, d\phi$$

$$= \frac{-1}{2\pi n^2} \int_{-\pi}^{\pi} f''(\phi) e^{-in\phi}\, d\phi$$

Since f'' is continuous on the circle, it is bounded, say by M. We obtain these bounds on the Fourier coefficients of f:

$$|\hat{f}(n)| \le \frac{M}{2\pi n^2}$$

Thus $\sum |f(n)| < \infty$.

Corollary 3. *If f is a C^2 function on the circle, it is the sum of its Fourier series.*

We shall have an even better result in Section 6.4. Nevertheless, Theorem 6.1 does allow us to make deductions on the convergence of the Fourier series. As one last application, it tells us that although we may not be able to approximate a function by its Fourier series, we can nevertheless approximate it by some sequence of trigonometric polynomials.

Corollary 4. *A continuous function on the circle is approximable by trigonometric polynomials.*

Proof. Using the notion of uniform continuity, we can be sure, in the proof of Theorem 6.1, that the δ chosen so that (6.13) is true is independent of θ. Thus, in the rest of the argument we find an $r < 1$ such that

$$|Pf(r, \theta) - f(\theta)| < \varepsilon \qquad \text{for all } \theta$$

Now, the series $\sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta}$ converges uniformly to $Pf(r, \theta)$, if $r < 1$. Thus there is an N such that the partial sum Q of the terms between $-N$ and N is everywhere within ε of $Pf(r, \theta)$. Thus

$$|Q(\theta) - f(\theta)| \leq |Q(\theta) - Pf(r, \theta)| + |Pf(r, \theta) - f(\theta)| \leq \varepsilon + \varepsilon = 2\varepsilon$$

for all θ, as desired.

• EXERCISES

1. Find the Fourier series of the following functions on the circle.
   (a) $f(\theta) = \theta^2$ (b) $f(\theta) = \cos^5 \theta$
   (c) $f(\theta) = e^{i\mu}$ $\mu > 0$, not necessarily an integer.

   (d)
$$f(\theta) = \begin{cases} 0 & -\pi \leq \theta \leq -\dfrac{\pi}{2} \\ \theta + \dfrac{\pi}{2} & -\dfrac{\pi}{2} \leq \theta \leq 0 \\ -\theta + \dfrac{\pi}{2} & 0 \leq \theta \leq \dfrac{\pi}{2} \\ 0 & \dfrac{\pi}{2} \leq \theta \leq \pi \end{cases}$$

   (e) $f(\theta) = |\sin \theta|$

(f) $f(\theta) = \sin\theta + \cos\theta$

(g)
$$f(\theta) = \begin{cases} 0 & -\pi \le \theta \le -\dfrac{\pi}{2} \\ 1 & -\dfrac{\pi}{2} \le \theta \le 0 \\ 0 & 0 \le \theta \le \dfrac{\pi}{2} \\ 1 & \dfrac{\pi}{2} \le \theta \le \pi \end{cases}$$

(h) $f(\theta) = e^{\theta}$
(i) $f(\theta) = e^{|\theta|}$

2. Find the Poisson transforms of the following functions on the circle:
   (a) $\cos^3\theta + \sin^3\theta$
   (b) $(1 + \cos^2\theta)^{-1}$
   (c) Exercise 1(c).
   (d) Exercise 1(d).
   (e) Exercise 1(g).
   (f) $(1 + e^{i\theta})^2$

• PROBLEMS

1. Show that the Fourier coefficients $\hat{f}(n)$ of a continuous function f defined on the circle are bounded:

$$|\hat{f}(n)| \le \|f\| = \max\{|f(\theta)| : -\pi \le \theta \le \pi\}$$

2. Show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r, t)\, dt = 1$$

by computing the Poisson transform of the function 1.

3. Show that if f is a real-valued function on the circle, $\hat{f}(-n) = \hat{f}(n)^{-}$.

4. (a) Show that the Poisson transform of f can be written

$$Pf(r, \theta) = \hat{f}(0) + \sum_{n=1}^{\infty} (\hat{f}(-n)\bar{z}^{-n} + \hat{f}(n)z^{n})$$

(b) Show that if $\hat{f}(-n) = 0$, $n > 0$, then Pf is the sum of a convergent power series in the unit disk.

(c) Show that if f can be written in the form

$$f(\theta) = F(e^{i\theta})$$

where F can be written as a convergent series in powers of z, $\bar{z}$, then

$$Pf(z) = F(z)$$

5. What is the Poisson transform of these functions?
   (a) $\exp(e^{i\theta})$ (d) $(1 + \cos^2\theta)^{-1}$
   (b) $(1 + 2z)^{-1}$ (e) $\ln(5 + z)$
   (c) $(z + \bar{z})^n$ (f) $\exp(\cos\theta)$
6. We can use the approximation theorem (Corollary 4) to prove the following fact.

(Weierstrass Approximation Theorem). *If f is a continuous function on the interval* $[0, 1]$ *and* $\varepsilon > 0$, *there is a polynomial* $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ *such that*

$$|f(x) - P(x)| < 1 \qquad \text{for all } x \in [0, 1]$$

Prove it according to this idea: First extend f as a continuous function on the interval $[-\pi, \pi]$ so that $f(-\pi) = f(\pi)$. Now view the extended function as a function on the circle and, by Corollary 4, approximate it by a trigonometric polynomial of the form $\sum_{n=-N}^{N} \alpha_n e^{in\theta}$. Now use the fact that the $2N$ functions $\{e^{in\theta}: -N \le n \le N\}$ can be approximated by polynomials in θ.

6.2 Laplace's Equation

The techniques described in the previous section came out of Poisson's work on the theory of heat flow. Suppose D is a domain in the plane (representing a homogeneous metallic plate); we wish to study the temperature distribution on this plate subject to certain sources of heat energy. Let $u(x, t)$ be the temperature at the point x at time t. We shall see in Chapter 8 that, as a consequence of the law of energy conservation, the temperature function u behaves according to this partial differential equation (appropriately called the **heat equation**):

$$\frac{\partial u}{\partial t} = \frac{1}{a^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{6.14}$$

Now, suppose our sources of heat maintain the temperature at the boundary of D, and there is no other source or loss of heat. Then, as $t \to \infty$ the temperature distribution will tend toward equilibrium: that state at which $\partial u/\partial t = 0$. This equilibrium (or steady-state) temperature distribution must therefore satisfy **Laplace's equation**:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{6.15}$$

This is sometimes denoted $\Delta u = 0$. Solutions of Laplace's equation are called **harmonic functions.**

The Poisson transform has to do with the solution of this steady-state problem when D is the unit disk. Suppose then, that we are given a temperature distribution $f(\theta)$ on the unit circle; we wish to find a continuous function $u(r, \theta)$ defined for $r \leq 1$ such that $\Delta u = 0$ and $u(1, \theta) = f(\theta)$. In order to attack this problem, we assume that u can be represented, on each circle $r =$ constant by its Fourier series:

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} a_n(r)e^{in\theta} \tag{6.16}$$

Our conditions become

$$\text{(i)} \quad a_n(1) = \hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\, d\theta \tag{6.17}$$

$$\text{(ii)} \quad \Delta u = r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \theta^2} = 0 \tag{6.18}$$

(We have rewritten Δu in terms of polar coordinates so we can apply it to the Fourier series. We leave it to the reader to derive the polar form of the Laplacian.)

Now, computing (ii) term by term in the series (6.16), we obtain

$$0 = \sum_{n=-\infty}^{\infty} (r^2 a_n'' + r a_n' - n^2 a_n)e^{in\theta} = 0$$

Since the zero function is represented only by the zero Fourier series we deduce that

$$r^2 a_n'' + r a_n' - n^2 a_n = 0 \tag{6.19}$$

for all n. This ordinary differential equation is easily solved:

$$\begin{array}{lll} 1, & \log r & n = 0 \\ r^n, & r^{-n} & n \neq 0 \end{array}$$

We have only one boundary condition (6.17), however we do want the functions continuous at $r = 0$, so the solutions $\log r$, $r^{-|n|}$ are excluded. Thus we must have $a_n = \hat{f}(n)r^{|n|}$, and the solution must have the form

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta}$$

which is Poisson's transform. Hence, if the problem is solvable, the solution must be given by Poisson's transform. Conversely, the following is a solution.

Theorem 6.2. *Let f be a continuous function on the circle. There is a unique function u, harmonic in the disk and assuming the boundary values f. u is the Poisson transform of f:*

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) \frac{1 - r^2}{1 + r^2 - 2 \cos(\theta - \phi)} \, d\phi$$

Proof. We need only verify that u is indeed harmonic. Since we can differentiate under the integral sign

$$\Delta u = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) \Delta \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - \phi)} \, d\phi$$

we need only show that the Poisson kernel is harmonic. That can be done by direct computation, or by referring back to Equation (6.9). There we have

$$P(r, \theta) = \frac{1}{1 - re^{-i\theta}} - 1 + \frac{1}{1 - re^{i\theta}} = \frac{1}{1 - \bar{z}} - 1 + \frac{1}{1 - z}$$

$$= -1 + 2 \operatorname{Re}\left(\frac{1}{1 - z}\right)$$

Now, $(1 - z)^{-1}$ is a complex differentiable function, and we have already seen that the real part of such a function satisfies Laplace's equation, see Problem 5.7.2 Thus $\Delta P = 0$.

To recapitulate, Laplace's equation for the disk with given boundary values is easily solved by Fourier methods. If f is the boundary temperature distribution, the solution is

$$u(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta} = \hat{f}(0) + \sum_{n=1}^{\infty} (\hat{f}(-n)\bar{z}^n + \hat{f}(n) z^n)$$

(since $r^{|n|}e^{in\theta} = z^n$ for $n > 0$, $r^{|n|}e^{in\theta} = \bar{z}^n$ for $n < 0$).

Examples

8. Find the solution of Laplace's equation with boundary values $f(\theta) = \cos^3\theta + 3\sin 3\theta$. This is easy to do, for we can easily recognize the function as the boundary value of the real part of a complex differentiable function. Since

$$\cos\theta = \frac{1}{2}(z + \bar{z}) \qquad \sin 3\theta = \frac{1}{2i}(z^3 - \bar{z}^3)$$

on the unit circle, we have

$$f(\theta) = \frac{(z + \bar{z})^3}{8} + \frac{3}{2i}(z^3 - \bar{z}^3)$$

for $z = e^{i\theta}$. Thus the solution is given by the same expression for all z, $|z| \le 1$ since it is clearly harmonic.

9. Solve Laplace's equation with boundary values $f(\theta) = |\theta|$. Since $|\theta|$ is *not* a trigonometric polynomial, we must compute the Fourier expansion.

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\theta| e^{-in\theta}\, d\theta = \frac{-1}{2\pi}\int_{-\pi}^{0} \theta e^{-in\theta}\, d\theta + \frac{1}{2\pi}\int_{0}^{\pi} \theta e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{\pi} \theta(e^{in\theta} + e^{-in\theta})\, d\theta = \frac{1}{\pi}\int_0^{\pi} \theta\cos n\theta\, d\theta$$

$$= \frac{1}{\pi n}\int_0^{\pi} \sin n\theta\, d\theta = \frac{1}{\pi n^2}\cos n\theta\Big|_0^{\pi} = \frac{-2}{\pi n^2} \qquad \begin{cases} n \text{ odd} \\ n \neq 0 \end{cases}$$

$$\hat{f}(0) = \frac{1}{\pi}\int_0^{\pi} \theta\, d\theta = \frac{\pi}{2}$$

Finally,

$$Pf(z) = \frac{\pi}{2} = \frac{2}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\bar{z}^n + z^n}{n^2} = \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{z^{2n+1} + \bar{z}^{2n+1}}{(2n+1)^2}$$

The problem analogous to the above in the case of a general domain is known as **Dirichlet's problem.** More precisely, Dirichlet's problem is to find for a given domain D and function f defined on D, a function harmonic in D and taking the given boundary values. In 1931, O. Perron gave an elementary, but extremely clever argument which proved the existence of a solution to Dirichlet's problem. Poisson's method plays a strategic role in Perron's arguments, which we shall not go into here. However, we shall verify that the solution is unique: there can be at most one harmonic function with given boundary values. This follows from the mean value property of harmonic functions.

Proposition 2. *Suppose u is a harmonic function in the domain D. If $\bar{\Delta}(z_0, R) \subset D$, then*

$$u(z_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(z_0 + Re^{i\theta})\, d\theta$$

that is, $u(z_0)$ is the average of its values around any circle in D contained in D.

Proof. We can expand u in a Fourier series around any circle $|z - z_0| = r$, $r \leq R$:

$$u(z) = \sum_{n=-\infty}^{\infty} a_n(r) e^{in\theta} \qquad r \leq R \qquad \text{where } z = z_0 + re^{i\theta}$$

Since $\Delta u = 0$, we must have $a_n(r) = \hat{f}(n) r^{|n|}$, where $f(\theta)) = u(z_0 + Re^{i\theta})$, already seen. Thus

$$u(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)\, |z - z_0|^{|n|} e^{in \arg(z - z_0)}$$

so

$$u(z_0) = \hat{f}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(z_0 + Re^{i\theta})\, d\theta$$

Corollary 1. *Suppose u is harmonic on the closed and bounded domain D. If $u > 0$ on ∂D, then $u > 0$ throughout D.*

Proof. Let us suppose that the conclusion is false. That at some point z_0 inside D, $u(z_0) \leq 0$. We shall derive a contradiction. We may take for z_0 a point at which u takes its minimum value. There is such a point since D is closed and bounded, and it is interior to D since $u > 0$ on ∂D. Let $\bar{\Delta}(z_0, R)$ be the largest disk centered at z_0 contained in D. The boundary of $\bar{\Delta}(z_0, R)$ must touch ∂D (see Figure 6.2), for if not we could find a larger disk centered at z_0 and contained in D. Thus there are points on the circle $|z - z_0| = R$ at which $u > 0$. Since $u(z_0)$ is the average value of u on this circle, and $u(z_0) \leq 0$, there must be points on this circle at which $u < u(z_0)$ in order to compensate. But $u(z_0)$ is the minimum value of u, so we have a contradiction. More precisely, since $u(z) \geq u(z_0)$ for all $z \in D$, $u(z_0 + Re^{i\theta}) - u(z_0) \geq 0$ for all θ. On the other hand, by the mean value property

$$\int_{-\pi}^{\pi} (u(z_0 + Re^{i\theta}) - u(z_0))\, d\theta = 0$$

When the integral of a continuous nonnegative function is zero, that function is identically zero. Thus,

$$u(z_0 + Re^{i\theta}) = u(z_0) \qquad \text{for all } \theta$$

This contradicts the fact that for some θ, $u(z_0 + Re^{i\theta}) > 0$.

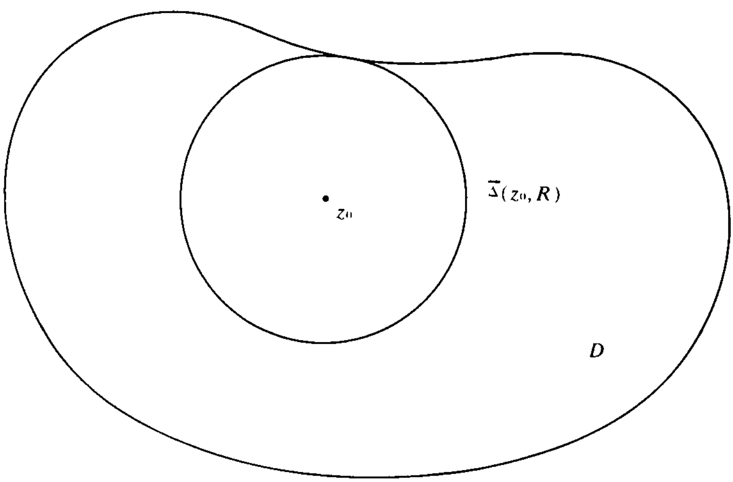

Figure 6.2

Corollary 2. *A function harmonic on a closed and bounded domain D is uniquely determined by its boundary values.*

Proof. Suppose that u, v are both harmonic in D, but $u = v$ on ∂D. Let $\varepsilon > 0$. Then $u - v + \varepsilon$, $v - u + \varepsilon$ are both positive on ∂D. By Corollary 2, they are both positive in D, thus

$$u \geq v - \varepsilon \qquad v \geq u - \varepsilon \qquad \text{in } D$$

Since ε is arbitrary, we may now let it tend to zero. We conclude that $u \geq v$ and $v \geq u$ throughout D. Thus $u = v$ in D.

Another problem of heat transfer is this: find the steady-state temperature distribution on the unit disk assuming a given rate of heat flow through the boundary, and no other source or loss of heat. Now the velocity of heat flow, denoted $\mathbf{q}$, is a vector field on the domain and it is a law of thermodynamics that this field is proportional to the temperature gradient, but oppositely directed. Thus, in this problem, our given data are the rate of heat flow perpendicular to the boundary of the unit disk, which is proportional to $\partial u/\partial r$ on the boundary. By the law of conservation of energy, since we are assuming a steady state, the total energy change is zero, thus we must impose this condition: $\int_{-\pi}^{\pi} \partial u/\partial r(e^{i\theta})\, d\theta = 0$. Thus, the mathematical formulation of this problem (known as **Neumann's problem**) is this: Find a function u harmonic in the unit disk such that $\partial u/\partial r(e^{i\theta})$ assumes given boundary values $g(\theta)$. We impose the condition $\int_{-\pi}^{\pi} g(\theta)\, d\theta = 0$. (It is necessary to impose this condition in order to obtain a solution, for mathematical reasons, as you will see in Problem 8.) We again solve this problem by Fourier methods. Find

$$u(re^{i\theta}) = \sum_{-\infty}^{\infty} a_n(r) e^{in\theta}$$

so that (i) $(\partial u/\partial r)(e^{i\theta}) = g(\theta)$, $\Delta u = 0$. Again, this leads to the ordinary differential equation (6.19) with the boundary condition $a_n'(1) = \hat{g}(n)$. The solution continuous at the origin is $|n|^{-1}\hat{g}(n)r^{|n|}$. Thus the solution must be given by

$$u(re^{i\theta}) = \sum_{-\infty}^{\infty} \frac{\hat{g}(n)}{n} r^{|n|} e^{in\theta} \tag{6.20}$$

We will omit the proof that this function does solve Neumann's problem; the argument is much like that in Theorem 6.1. We can, of course, collapse

(6.20) into an integral formula:

$$u(re^{i\theta}) = \sum_{-\infty}^{\infty} \frac{1}{n}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\phi)e^{-in\phi}\,d\phi\right] r^{|n|}e^{in\theta}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\phi)\left[\sum_{n=1}^{\infty} \frac{r^n e^{-in(\theta-\phi)}}{n} + \sum_{n=1}^{\infty} \frac{r^n e^{in(\theta-\phi)}}{n}\right] d\phi$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} g(\phi)\,\mathrm{Re}\left[\sum_{n=1}^{\infty} \frac{(re^{i(\theta-\phi)})^n}{n}\right] d\phi$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} g(\phi)\,\mathrm{Re}[\ln(1 - \mathrm{r}e^{i(\theta-\phi)})]\,d\phi$$

Now

$$|1 - re^{it}|^2 = 1 + r^2 - 2\cos t$$

so

$$\mathrm{Re}\ln(1 - re^{it}) = \tfrac{1}{2}\ln|1 - re^{it}|^2 = \tfrac{1}{2}\ln(1 + r^2 - 2r\cos t)$$

Thus the solution to Neumann's problem takes the form (6.20) or

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\phi)\ln[1 + r^2 - 2r\cos(\theta - \phi)]\,d\phi$$

• EXERCISES

3. Solve Dirichlet's problem in the disk with these boundary conditions:
   (a) $f(\theta) = \begin{cases} -1 & \theta < 0 \\ 1 & \theta > 0 \end{cases}$
   (b) $f(\theta) = \sin^2\theta - \cos^2\theta$
   (c) $f(\theta) = \pi^2 - \theta^2$
   (d) f as is given in Exercise 1(c).
   (e) $f(\theta)$ as is given in Exercise 2(f).
4. Solve Neumann's problem with these boundary conditions:
   (a) $f(\theta) = \sin\theta + 2\cos 2\theta$
   (b) $f(\theta) = \begin{cases} -\theta^2 & \theta \le 0 \\ \theta^2 & \theta \ge 0 \end{cases}$
   (c) f as is given in Exercise 3(a).

• PROBLEMS

7. Show that the Laplacian is given in polar coordinates by

$$\Delta u = r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \theta^2}$$

8. Verify that it is necessary that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\, d\theta = 0$$

for there to be a function u harmonic in the disk such that

$$\lim_{r\to 1}\frac{\partial u}{\partial r}(r, \theta) = g(\theta)$$

9. Verify by direct computation that $P(r, \theta)$ is harmonic.

10. Show that if f is a complex differentiable function (it satisfies the Cauchy-Riemann equations), then f is harmonic.

11. We can prove, using the Poisson transform, this remarkable fact about complex differentiable functions:

Theorem. *Suppose that f is a complex differentiable function on the unit disk. Then f is the sum of a convergent power series centered at the origin.*

The proof goes like this: Let $g(\theta) = f(e^{i\theta})$. Since f is harmonic in the disk (Problem 10), it solves Dirichlet's problem with the boundary values g. Thus $f(re^{i\theta}) = Pg(re^{i\theta})$. Now prove this fact.

(a) If the Poisson transform Pg is complex differentiable, then $\hat{g}(n) = 0$ for $n < 0$. (*Hint*: Apply $\partial/\partial x + i\,\partial/\partial y$ to the expression

$$Pg(re^{i\theta}) = \hat{g}(0) + \sum_{n=1}^{\infty} (\hat{g}(-n)\bar{z}^n + \hat{g}(n)z^n)$$

(b) Deduce from (a) that

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} \hat{g}(n)z^n$$

12. Under what conditions on f, g is $P(fg) = P(f)P(g)$?

13. (a) Show that if f is a continous function on the domain D with the mean value property:

$$f(z_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z_0 + Re^{i\theta})\, d\theta \qquad \text{for every } \bar{\Delta}(z_0, R) \subset D$$

then f satisfies a maximum principle: $f(z_0) \leq \max\{f(z): z \in \partial D\}$, for every $z_0 \in D$.

(b) Conclude that a function having the mean value property is harmonic.

14. Prove: A bounded function defined on the entire plane which is harmonic, must be constant.

6.3 Fourier Sine and Cosine Series

There are many notationally different ways of expressing the Fourier expansion of a function, depending mostly on the dictates of the problem at hand. We shall devote this section to the development of these various expressions.

First of all, since the main physical study is that of real-valued functions we should introduce the purely real notation. We merely convert the Fourier expansion $\sum \hat{f}(n)e^{in\theta}$ via the expressions

$$e^{in\theta} = \cos n\theta + i \sin n\theta \qquad e^{-in\theta} = \cos n\theta - i \sin n\theta \qquad n \geq 0$$

Thus the Fourier expansion will take the form

$$A_0 + \sum_{n=0}^{\infty} A_n \cos n\theta + B_n \sin n\theta \tag{6.21}$$

where the A's and B's are found from the Fourier coefficients $C_n = \hat{f}(n)$ as follows:

$$\sum_{n=-\infty}^{\infty} C_n e^{in\theta} = \sum_{n=-\infty}^{\infty} C_n(\cos n\theta + i \sin n\theta)$$

$$= C_0 + \sum_{n=1}^{\infty} [(C_n + C_{-n})\cos n\theta + i(C_n - C_{-n}) \sin n\theta]$$

Thus

$$A_0 = C_0 \qquad A_n = C_n + C_{-n} \qquad B_n = i\{C_n - C_{-n}\} \qquad n > 0$$

Notice that if f is real valued

$$C_{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi)e^{in\varphi}\, d\phi = \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi)e^{-in\varphi}\, d\phi\right]^{-} = \bar{C}_n$$

Thus we have

$$A_0 = C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,d\phi$$

$$A_n = 2\,\mathrm{Re}\,C_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\cos n\phi\,d\phi \qquad n > 0 \tag{6.22}$$

$$B_n = 2\,\mathrm{Im}\,C_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\sin n\phi\,d\phi \qquad n > 0 \tag{6.23}$$

Furthermore,

$$C_n = \tfrac{1}{2}(A_n + iB_n) \qquad C_{-n} = \tfrac{1}{2}(A_n - iB_n) \qquad n > 0$$

Examples

10. Express the Fourier series of $\pi^2 - \theta^2$ in the form (6.21). From Example 4, we have

$$\pi^2 - \theta^2 = \frac{2\pi^2}{3} + 2\sum_{n\neq 0}\frac{(-1)^n}{n^2}e^{in\theta}$$

Thus

$$A_0 = \frac{2}{3}\pi^2 \qquad A_n = 4\frac{(-1)^n}{n^2} \qquad B_n = 0$$

and we obtain this Fourier expansion:

$$\pi^2 - \theta^2 = \frac{2\pi^2}{3} + 4\sum_{n>0}\frac{(-1)^n}{n^2}\cos n\theta$$

Notice that equality is justified by Corollary 1 to Theorem 6.1 since the Fourier series does converge. Evaluating at 0, we obtain this interesting fact

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{\pi^2}{12}$$

11. Express the Fourier series of $|\theta|$ in the form (6.21). Reading from Example 9, we have the Fourier series for $|\theta|$:

$$\frac{\pi}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{e^{-in\theta} + e^{in\theta}}{n^2}$$

Thus we have the real Fourier series

$$|\theta| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2}$$

Evaluating at $\theta = 0$, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

12. As usual, trigonometric polynomials can be handled directly, without computation of integrals:

$$\cos^4 \theta = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^4 = \frac{1}{16}(e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta})$$

$$= \frac{1}{16}(2 \cos 4\theta + 8 \cos 2\theta + 6)$$

$$\cos^4 \theta = \frac{3}{8} + \frac{1}{2} \cos 2\theta + \frac{1}{8} \cos 4\theta$$

Even and Odd Functions

A function of a real variable is called an **even function** if $f(x) = f(-x)$ for all x, and it is an **odd function** if $f(x) = -f(x)$. Notice that the product of two odd functions is even, and the product of an odd and even function is odd. If f is an odd function on the interval $[-A, A]$, then

$$\int_{-A}^{A} f(t)\, dt = \int_{-A}^{0} f(t)\, dt + \int_{0}^{A} f(t)\, dt = -\int_{0}^{A} f(t)\, dt + \int_{0}^{A} f(t)\, dt = 0$$

We can conclude that if f is an even function on the interval $[-\pi, \pi]$, its Fourier series is purely a cosine series. For in this case $f(\phi) \sin n\phi$ is odd for all n, so the integrals (6.23) all vanish. Similarly, if f is an odd function its Fourier series is purely a sine series.

Example

13. The Fourier series of θ is of the form

$$\sum_{n=1}^{\infty} B_n \sin n\theta$$

since θ is an odd function. Here

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \theta \sin n\theta \, d\theta = -\frac{1}{\pi}\frac{\theta}{n}\cos n\theta\Big|_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos n\theta}{n}\, d\theta = -\frac{2}{n}(-1)^n$$

Thus θ has the Fourier series $-2\sum_{n=1}^{\infty}(-1)^n/n \sin n\theta$.

Now, all our computations have been done for periodic functions of period 2π. Periodic functions arising in physics do not usually have such a convenient period, yet they are subject to Fourier methods merely by a normalization. Suppose that f is a periodic function of period L. Then $g(\theta) = f(L\theta/2\pi)$ is periodic of period 2π. For

$$g(\theta + 2\pi) = f\left(\frac{L}{2\pi}(\theta + 2\pi)\right) = f\left(\frac{L\theta}{2\pi} + L\right) = f\left(\frac{L\theta}{2\pi}\right) = g(\theta)$$

Now, if g can be expanded in a Fourier series:

$$g(\theta) = A_0 + \sum_{n=1}^{\infty}(A_n \cos n\theta + B_n \sin n\theta)$$

then we can write

$$f(x) = g\left(\frac{2\pi x}{L}\right) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2\pi n x}{L}\right) + B_n \sin\left(\frac{2\pi n x}{L}\right) \tag{6.24}$$

where (as is easy to compute by the change of coordinates $\phi = 2L^{-1}x$)

$$A_0 = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, dx \qquad A_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x) \cos\frac{2\pi n x}{L}\, dx \tag{6.25}$$

$$B_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x) \sin\frac{2\pi n x}{L}\, dx \tag{6.26}$$

With these formulas the Fourier analysis of functions periodic of period L is made possible.

Fourier Cosine Series

There are yet two more variations which are, as we shall see, of value in the study of partial differential equations. Let f be a given periodic function with period L and define

$$\begin{aligned} g(\theta) &= f\left(\frac{L\theta}{\pi}\right) \qquad && 0 \le \theta \le \pi \\ &= f\left(\frac{-L\theta}{\pi}\right) \qquad && -\pi \le \theta \le 0 \end{aligned} \tag{6.27}$$

Then g is an even function on the interval $[-\pi, \pi]$, so it can be expressed by a Fourier series involving only cosines:

$$g(\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta$$

where

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\, d\theta \qquad & A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta) \cos n\theta \, d\theta \\ &= \frac{1}{\pi}\int_0^{\pi} g(\theta)\, d\theta & &= \frac{2}{\pi}\int_0^{\pi} g(\theta)\cos n\theta\, d\theta \end{aligned}$$

Now, making the substitution $g(\theta) = f(L\theta/\pi)$ in the interval $0 \le \theta \le \pi$, these expressions become

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{\pi n x}{L} \tag{6.28}$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\, dx \qquad A_n = \frac{2}{L}\int_0^L f(x) \cos \frac{\pi n x}{L}\, dx \tag{6.29}$$

We pause to remind the reader that the use of equality in Equations (6.24) and (6.28) is not literal, it holds only if the series converge (say if g is twice continuously differentiable). The point is that in such cases the expansions (6.24), (6.28) are valid, where the coefficients are defined by (6.25), (6.26), or

(6.29), respectively. The choice of these expansions is free—it is usually dependent on the demands of the particular problem at hand. Equation (6.28) is called the **Fourier cosine series** for the function f. Of course, if we define g as an odd function, instead of the expression (6.28) we can obtain the Fourier sine series for f:

$$f(x) = \sum_{k=1}^{\infty} B_n \sin \frac{\pi n x}{L} \tag{6.30}$$

where

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{\pi n x}{L} \, dx \tag{6.31}$$

We leave the verification of this possibility to the readers as a problem.

• EXERCISES

5. Find the Fourier expansions into sines and cosines for these functions:
   (a) $\cos^8 \theta$
   (b) $\sin^k \theta$ k a positive integer
   (c) f as given in Exercise 3(a).
   (d) f as given in Exercise 1(g).
   (e) f as given in Exercise 1(b).
6. Find the function whose Fourier expansion is $\sum_{n=-\infty}^{\infty} e^{in\theta}/in$.
7. Find the Fourier sine and Fourier cosine series for these periodic functions of period 1.
   (a) $f(x) = 1$, all x
   (b) $f(x) = \sin(2\pi x)$
   (c) $f(x) = \cos(2\pi x)$
   (d) $f(x) = \begin{cases} 1 & 0 \le x \le 1/2 \\ 0 & 1/2 \le x \le 1 \end{cases}$
   (e) $f(x) = \sin(\pi x)$
   (f) $f(x) = \begin{cases} x & 0 \le x \le 1/2 \\ 1 - x & 1/2 \le x \le 1 \end{cases}$
   (g) $f(x) = \sin(\pi x) + \cos(\pi x)$
8. Show that any periodic function on the circle is the sum of an even function and an odd function.
9. What is the Fourier expansion of $f(\theta) + f(\pi - \theta)$ in terms of that for $f(\theta)$?

6.4 The One-Dimensional Wave and Heat Equations

In physics, Fourier analysis begins with the study of wave motions. Suppose we have a homogeneous string of density ρ and length L lying on the horizontal axis in the plane which is kept extended by equal and opposite forces of magnitude k at the end points. If we pluck the string, it will follow a motion which is (classically) determined by Newton's laws. We shall derive the differential equation governing the motion. At some time t the string has a shape somewhat like that pictured in Figure 6.3. We shall refer to a point on the string according to the distance s, measured along the string from the left end point. The position in the plane of the point at distance s at time t will be denoted by $\mathbf{z}(s, t)$. This is the function that fully describes the motion.

Now, if we argue as if the string were a collection of points, we will get nowhere. For the only forces acting on the string are those obtained by transferring the equal, but opposite forces at the end points tangentially along the string. Thus, at any point the sum of the forces acting is zero, so there can be no motion. As that is contrary to fact, this model of the string is inadequate and we must select another.

Now we consider the string as a large finite collection of segments and again try to deduce the equation of motion from Newton's laws. Having done that, we can idealize by letting the number of segments become infinite (as their lengths tend to zero) and obtain a differential equation. Let s_0 and $s_0 + \Delta s$ be the end points of such a segment (see Figure 6.4). The mass of this segment is $\rho \Delta s$ and the forces acting on it are opposed tangential forces of magnitude k acting at the end points. Letting $\mathbf{T}(s)$ be the tangent vector at the point s, these forces are thus $-k\mathbf{T}(s_0)$, $k\mathbf{T}(s_0 + \Delta s)$, respectively. If $\mathbf{A}$ is the acceleration of this segment, we have by Newton's laws

$$\rho \Delta s \mathbf{A} = k[\mathbf{T}(s_0 + \Delta s) - \mathbf{T}(s_0)]$$

Now, $\mathbf{T}(s) = \partial \mathbf{z}(s, t)/\partial s$ and $\lim_{\Delta s \to 0} \mathbf{A} = \partial \mathbf{z}/\partial t(s_0, t)$. Thus

$$\mathbf{A} = \frac{k}{\rho} \frac{(\partial \mathbf{z}/\partial s)(s_0 + \Delta s, t) - (\partial \mathbf{z}/\partial s)(s_0, t)}{\Delta s}$$

and now letting $\Delta s \to 0$ we obtain the equation of motion:

$$\frac{\partial^2 \mathbf{z}}{\partial t^2}(s_0, t) = \frac{k}{\rho} \frac{\partial^2 \mathbf{z}}{\partial s^2}(s_0, t)$$

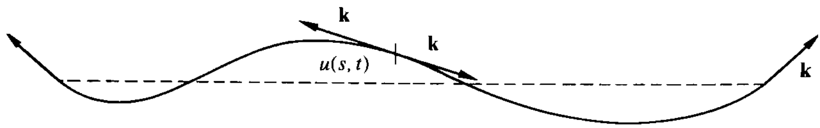

Figure 6.3

This equation, called the one-dimensional wave equation, is usually written

$$\frac{\partial^2 \mathbf{z}}{\partial s^2} = \frac{1}{c^2} \frac{\partial^2 \mathbf{z}}{\partial t^2} \tag{6.32}$$

(where the substitution $c^2 = k/\rho$ is legitimate since both k, ρ are positive). We now make the (physically plausible) assumption that the horizontal motion is negligible (for we are interested only in almost horizontal wave motions with small fluctuations). This assumption allows the replacement of s by the horizontal coordinate x, and the positive vector $\mathbf{z}$ by only the vertical coordinate y. Thus (6.32) becomes simply

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} \tag{6.33}$$

The motion of the string is completely governed by this partial differential equation and the initial displacement and velocity:

$$y(s, 0) = f(s) \qquad \frac{\partial y}{\partial t}(s, 0) = g(s) \tag{6.34}$$

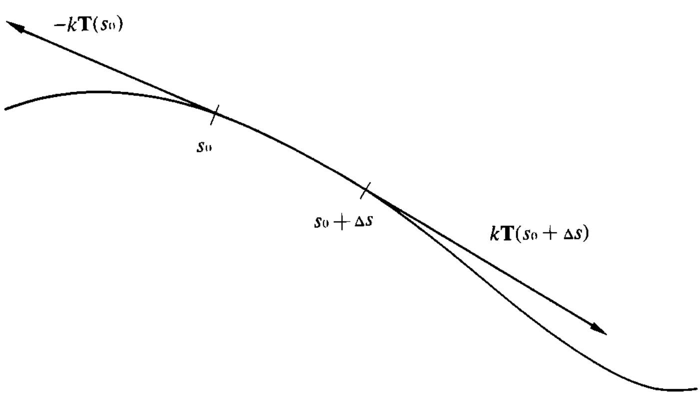

Figure 6.4

The technique for solving this differential equation with boundary conditions is the same as in the theory of ordinary differential equations. We find an independent set of solutions of the general equations and hypothesize that the solution we seek is a linear combination of these. We then identify the coefficients by substituting the initial conditions. However, the situation is more complicated than in the one-variable theory. The space of solutions of (6.32) is infinite dimensional, so the particular solution cannot be picked out of the general solution by means of simple linear algebra. This difficulty will be overcome, as we shall see, because the form of the general solution will be that of a Fourier expansion and so the initial data will give us the coefficients by Fourier methods.

Let us now solve the differential equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \tag{6.35}$$

for a function y defined on the interval $[0, L]$ and where these conditions must be satisfied

$$y(0, t) = 0 \qquad y(L, t) = 0 \qquad \text{all } t \tag{6.36}$$

$$y(x, 0) = f(x) \qquad \frac{\partial y}{\partial t}(x, 0) = g(x) \tag{6.37}$$

for given functions f, g. First, we put aside the initial data (6.37) and find all solutions of Equation (6.35) subject to (6.36). Since we have no techniques available, we have to make a guess at the form of the solution, and hope that our guess is general enough (of course, in the end it will turn out to be so). The guess that works is

$$y(x, t) = F(x)G(t)$$

and (6.35) becomes

$$F''(x)G(t) = \frac{1}{c^2}F(x)G''(t)$$

or, what is the same (since we exclude the zero solution),

$$\frac{F''(x)}{F(x)} = \frac{1}{c^2}\frac{G''(t)}{G(t)}$$

The left-hand side is independent of t, and the right is independent of x. Since they are the same, they are both constant. Thus, there must be a λ such that

$$\frac{F''}{F} = \lambda \qquad \frac{1}{c^2}\frac{G''}{G} = \lambda$$

Now, incorporating the conditions (6.36), we arrive at this one-variable boundary value problem:

$$F'' - \lambda F = 0 \qquad \text{for some } \lambda \tag{6.38}$$

$$F(0) = 0 \qquad F(L) = 0 \tag{6.39}$$

We *can* find all solutions of this problem. First of all, we see from (6.38) that the general form of F is

$$F(x) = c_1 \exp(\sqrt{\lambda} x) + c_2 \exp(-\sqrt{\lambda} x)$$

Substituting the boundary conditions (6.39), we have

$$0 = F(0) = c_1 + c_2 \qquad 0 = F(L) = c_1 \exp(\sqrt{\lambda} L) + c_2 \exp(-\sqrt{\lambda} L)$$

In order for there to be a solution for both equations we must have $c_1 = -c_2$ and

$$\exp(\sqrt{\lambda} L) = \exp(-\sqrt{\lambda} L) \quad \text{or} \quad \exp(2\sqrt{\lambda} L) = 1$$

Thus we must have $2\sqrt{\lambda} L = 2\pi n i$ for some $n \geq 0$, or $\sqrt{\lambda} = \pi n i/L$. Therefore, the only possible solutions of (6.38), (6.39) are

$$F(x) = \exp\left(\frac{\pi n i}{L}\right)x - \exp\left(-\frac{\pi n i}{L}\right)x = 2i \sin\left(\frac{\pi n}{L} x\right) \qquad \text{all } n \geq 0$$

Corresponding to the solution $F_n(x) = \sin(\pi n/L)x$, we now solve for G:

$$G'' = -\frac{\pi^2 n^2}{L^2 c^2} G$$

The solutions are spanned by $G_n(t) = \cos(\pi n/Lc)t$, $\sin(\pi n/Lc)t$. Thus, all solutions of (6.35) of the form $F(x)G(t)$ are these:

$$\sin\left(\frac{\pi n x}{L}\right)\cos\left(\frac{\pi n}{Lc}t\right) \qquad \sin\left(\frac{\pi n x}{L}\right)\sin\left(\frac{\pi n}{Lc}t\right) \tag{6.40}$$

We now return to our particular initial conditions (6.37) and hope to find a linear combination of the functions (6.40) which has those initial conditions. Of course, the linear combination will satisfy (6.37) since it is a linear differential equation. (However, we must caution the reader that ours will be an infinite linear combination so questions of convergence are inevitable. If the initial data are well behaved, these problems disperse as you shall see in Problem 15.) Thus we seek

$$y(x, t) = \sum_{n=0}^{\infty}\left[A_n \cos\left(\frac{\pi n}{Lc}t\right) + B_n\left(\sin\frac{\pi n}{Lc}t\right)\right]\sin\left(\frac{\pi n}{L}x\right)$$

satisfying the conditions (6.37):

$$f(x) = y(x, 0) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{\pi n}{L}x\right)$$

$$g(x) = \frac{\partial y}{\partial t}(x, 0) = \sum_{n=0}^{\infty} B_n \frac{\pi n}{Lc}\sin\left(\frac{\pi n}{L}x\right)$$

But we can solve these equations, for these are just the expansions of f and g into Fourier sine series. We collect this discussion into the following proposition.

Proposition 3. *If the functions f, g defined on the interval $[0, L]$ are well behaved (say at least twice differentiable), then the wave equation*

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

with the boundary data $y(0, t) = 0$, $y(L, t) = 0$ and the initial data $y(x, 0) = f(x)$, $(\partial y/\partial t)(x, 0) = g(x)$ has a solution. The solution is given by

$$y(x, t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{\pi n t}{Lc}\right) + B_n \sin\left(\frac{\pi n t}{Lc}\right)\right]\sin\left(\frac{\pi n x}{L}\right)$$

where

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{\pi n x}{L}\right)dx \qquad B_n = \frac{2c}{n}\int_0^L g(x)\sin\left(\frac{\pi n x}{L}\right)dx$$

Examples

14. Solve the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{4}\frac{\partial^2 x}{\partial t^2}$$

on the interval $[0, \pi]$ with initial data

$$y(x, 0) = \sin 2x \qquad (\partial y/\partial t)(x, 0) = \sin^2 x$$

Now $c = 2$, $L = \pi$. The Fourier sine series for $y(x, 0)$ is just $\sin 2x$, so $A_n = 0$ unless $n = 2$, $A_2 = 1$. Now

$$B_n = \frac{4}{\pi n}\int_0^\pi \sin^2 x \sin(nx)\,dx = \frac{4}{\pi n}\int_0^\pi \frac{1 - \cos 2x}{2}\sin(nx)\,dx$$

$$= 0 \quad \text{if } n \text{ is even}$$

We concentrate now on the case where n is odd:

$$B_n = \frac{2}{\pi n}\cdot\frac{2}{n} - \frac{2}{\pi n}\int_0^\pi \cos(2x)\sin(nx)\,dx \tag{6.41}$$

Now

$$\int_0^\pi \cos(2x)\sin(nx)\,dx$$

$$= -\left.\frac{\cos(2x)\cos(nx)}{n}\right|_0^\pi - \frac{2}{n}\left.\frac{\sin(2x)\sin(nx)}{n}\right|_0^\pi$$

$$+ \frac{4}{n^2}\int_0^\pi \cos(2x)\sin(nx)\,dx$$

Thus

$$\left(1 - \frac{4}{n^2}\right) \int_0^{\pi} \cos(2x)\sin(nx)\,dx = \frac{1}{n}(1 - \cos \pi n) = \frac{2}{n} \qquad (n \text{ odd})$$

Now, putting the result of this computation into (6.41):

$$B_n = \frac{2}{\pi n}\left[\frac{2}{n} - \frac{2n}{n^2 - 4}\right] = \frac{-16}{n^2(n^2 - 4)}$$

Thus the solution is given by

$$y(x, t) = \cos t \sin 2x - \frac{16}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin nt/2}{n^2(n^2 - 4)} \sin nx$$

$$y(x, t) = \cos t \sin 2x - \frac{16}{\pi} \sum_{m=1}^{\infty} \frac{\sin(mt)\sin 2mx}{(2m + 1)^2(4m^2 + 2m - 3)}$$

15. Solve the same wave equation with initial data

$$y(x, 0) = \sin x + \sin 5x + 2 \sin 6x \qquad \frac{\partial y}{\partial t}(x, 0) = 0$$

The expressions for the initial conditions are the Fourier sine series for those functions; thus we can read off the solution:

$$y(x, t) = \cos \frac{t}{2} \sin x + \cos \frac{5t}{2} \sin 5x + 2 \cos 3t \sin 6x$$

Heat Transfer

Another physical problem which gives rise to a partial differential equation which can be solved in a similar way is the problem of one-dimensional heat transfer. We shall derive this equation here (the derivation in Chapter 8 of this equation in higher dimensions shall be seen to be completely analogous). Suppose we are given a thin homogeneous rod of length L lying on the horizontal axis. Let $u(x, t)$ be the temperature at x at time t. We assume that there is no heat loss, and the temperatures at the end points are maintained constant. Now the basic physical law here is that the flow of heat is proportional to the temperature gradient, but points in the opposite direction.

Thus, during a small interval of time Δt the heat (energy) passing from left to right through a point x_0 is proportional to $-(\partial u/\partial x)(x_0) \cdot \Delta t$. If we select a segment of the rod with end points x_0 and $x_0 + \Delta x$ the increase in energy in that segment of the rod is proportional to

$$-\left(-\frac{\partial u}{\partial x}(x_0 + \Delta x)\,\Delta t\right) + \left(-\frac{\partial u}{\partial x}(x_0)\,\Delta t\right) \tag{6.42}$$

On the other hand, the increase in energy is proportional to the product of the mass and the change in temperature. Thus (6.42) is proportional to $\Delta u \cdot \Delta x$. Letting k^2 be the constant of proportionality we have, for the period of time Δt:

$$\Delta u \cdot \Delta x = k^2\left[\frac{\partial u}{\partial x}(x_0 + \Delta x) - \frac{\partial u}{\partial x}(x_0)\right]\Delta t$$

Dividing by $\Delta x \cdot \Delta t$ and letting both tend to zero, we obtain the **heat equation**:

$$\frac{1}{k^2}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{6.43}$$

We now propose to solve (6.43) given the boundary conditions

$$u(0, t) = 0 \qquad u(L, t) = 0 \tag{6.44}$$

and the initial temperature distribution

$$u(x, 0) = f(x) \tag{6.45}$$

The technique is the same as that for the wave equation. We try a solution of the form $u(x, t) = F(x)G(t)$. (6.43) becomes

$$G'(t)F(x) = \frac{1}{k^2}F''(x)G(t)$$

Dividing by $F(x)G(t)$, we again find that there must be a λ such that

$$\frac{F''}{F} = \lambda \qquad \frac{G'}{G} = \frac{\lambda}{k^2}$$

The first equation, subject to the initial conditions (6.44) again has only the solutions $\sin(\pi n x/L)$, $n \geq 0$, corresponding to the choices $\sqrt{\lambda} = \pi n i/L$. The second equation becomes

$$G' = \frac{-\pi^2 n^2}{L^2 k^2} G$$

which has the solutions

$$G_n(t) = \exp\left(\frac{-\pi^2 n^2}{L^2 k^2}\right) t$$

For convenience, let us write $C = \pi/Lk$. We now try to fit the series

$$\sum_{n=1}^{\infty} A_n \exp(-C^2 n^2 t) \sin\left(\frac{\pi n x}{L}\right) \tag{6.46}$$

to the initial conditions. Evaluating at $t = 0$ we find that the $\{A_n\}$ must be the Fourier sine coefficients of $f(x)$.

Proposition 4. *If the function f defined on the interval* $[0, L]$ *is well-behaved (say at least twice continuously differentiable), then the heat equation*

$$\frac{1}{k^2}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with the boundary data $y(0, t) = 0 = y(L, t)$ and the initial condition $u(x, 0) = f(x)$ has a solution. The solution is given by (6.46) *where $C = \pi/Lk$ and*

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{\pi n x}{L}\right) dx$$

Now, the wave and heat equations readily and conveniently led us to the considerations of Fourier analysis. Actually this could have been (and in fact was) anticipated on physical grounds, for we should expect periodic behavior in these circumstances. Other partial differential equations arising out of physics can be solved by similar techniques, but we do not necessarily end up with a sequence of solutions of the general equation which are made up of trigonometric functions. Thus the Fourier analysis does not apply, whereas the fundamental ideas may carry over. The typical situation is this: a partial differential operator P is given on a certain domain D; we seek a solution f of

$$P(f) = 0$$

subject to certain boundary conditions "B" and initial data $f(x, 0) = g(x)$. First, we find all solutions of $P(f) = 0$ subject to the boundary conditions B, without regard to initial conditions. If $\{S_1, \ldots, S_n, \ldots\}$ are these solutions, then we try to find a linear combination $\sum a_n S_n$ which fits our initial data:

$$\sum a_n S_n(x, 0) = g(x)$$

In our typical situation the $S_n(x, 0)$ are *orthonormal* in the sense of some *convenient* inner product on the space of all initial data. In this case the a_n are readily computable:

$$a_n = \langle S_n, g \rangle$$

The cases of the heat and wave equations described above are just special cases of this method. There are many more examples of such orthogonal expansions; discussions of them can be found in most texts of mathematical physics.

Finally, we cannot really expect to be able to follow through such a program for *every* partial differential equation, thus the general theory does not follow such an explicit line of reasoning. In one approach, local solutions are sought through examination of Taylor expansions (everything involved is assumed analytic). This is the Cauchy-Kowalewski theory. A more recent attack has its roots in the above ideas, as well as the Picard theorem. The vector space of differentiable functions is provided with a notion of *distance* and *length* which is suited to the given problem so that one can resolve questions of existence and uniqueness (as in the Picard theorem) and provide usable approximations with estimates derived from the initial data. This study is one of the most active branches of modern mathematics.

• EXERCISES

10. Solve the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 x}{\partial t^2}$$

on the interval (0, 1) with the boundary data $y(0, t) = 0 = y(1, t)$, and the following initial data

(a) $y(x, 0) = \sin x$ $\qquad\qquad \dfrac{\partial y}{\partial t}(x, 0) = 0$

(b) $y(x, 0) = \cos^3 \pi x - \cos \pi x$ $\qquad \dfrac{\partial y}{\partial t}(x, 0) = \sin \pi x$

(c) $y(x, 0) = x(x - 1)$ $\qquad \dfrac{\partial y}{\partial t}(x, 0) = 0$

(d) $y(x, 0) = \cos \pi x$ $\qquad \dfrac{\partial y}{\partial t}(x, 0) = \sin \pi x$

(e) $y(x, 0) = 0$ $\qquad \dfrac{\partial y}{\partial t}(x, 0) = \sin \dfrac{3\pi}{2} x + \sin \dfrac{\pi}{2} x$

11. Solve the heat equation

$$\frac{\partial u}{\partial t} = 4 \frac{\partial^2 u}{\partial x^2}$$

on the interval $(0, L)$ with the boundary data

$$u(0, t) = 0 = u(L, t)$$

and the following initial data

(a) $u(x, 0) = \sin x$

(b) $u(x, 0) = \cos \dfrac{\pi x}{L}$

(c) $u(x, 0) = x(x - L)$

(d) $u(x, 0) = \sin \dfrac{\pi x}{L} + 3 \sin \dfrac{5\pi x}{L}$

12. (a) Show that the function $u(x, t) = ax + b$ solves the heat equation on the interval $(0, L)$, with boundary data

$$u(0, t) = b, \ u(L, t) = aL + b$$

(b) Show that if u, v solve the heat equation with boundary data

$$u(0, t) = t_1 \qquad u(L, t) = t_2 \qquad v(0, t) = 0 \qquad v(L, t) = 0$$

then $u + v$ solves the heat equation with the same boundary data as u.

(c) Solve the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

on the interval $(0, 1)$ with boundary data $u(0, t) = 1$, $u(1, t) = e^t$ and initial data $u(x, 0) = e^x$.

13. The initial data given in the problem of heat flow may be the rate of flow of heat energy; or what is the same, the gradient of the temperature. Show that the solution of the heat equation

$$\frac{1}{k^2}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

on the interval $(0, L)$ with boundary data $u(L, 0) = 0 = u(L, t)$ and initial data $(\partial u/\partial x)(x, 0) = f(x)$ is given by

$$\sum_{n=1}^{\infty} A_n \exp(-C^2 n^2 t)\cos\frac{\pi n x}{L}$$

where C is a constant, and

$$A_n = \frac{2}{\pi n}\int_0^L f(x)\cos\frac{\pi n x}{L}\,dx$$

14. Solve the heat equation given in Exercise 11 with this initial data:
   (a) $\partial u/\partial x(x, 0) = \cos \pi x/L$,
   (b) $\partial u/\partial x(x, 0) = \sin \pi x/L$.

15. Solve the differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} + u$$

on the interval $(0, \pi)$ with boundary data $u(0, t) = 0 = u(1, t)$ and the initial data $u(x, 0) = f(x)$.

16. Do the same where the differential equation is

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial x}$$

• PROBLEMS

15. Show that the series defining the function $y(x, t)$ in Proposition 2 converges uniformly and absolutely under the stated conditions. Does this observation suffice to deduce the conclusion of Proposition 2?

16. We may be given, in the heat problem, the gradient of the temperature as boundary data. Show that the general solution of the heat equation with boundary data

$$\frac{\partial u}{\partial x}(0, t) = 0 = \frac{\partial u}{\partial x}(L, t)$$

can be written as a Fourier cosine series. Solve the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2}$$

on the interval $(0, \pi)$ with the boundary conditions

$$\frac{\partial u}{\partial x}(0, t) = 0 = \frac{\partial u}{\partial x}(L, t)$$

and the initial conditions

(a) $u(x, 0) = \sin x$

(b) $\dfrac{\partial u}{\partial x}(x, 0) = \sin x$

17. Solve the differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with the boundary data $\partial u/\partial x(0, t) = 0$, $\partial u/\partial x(L, t) = h$ and initial conditions $u(x, 0) = 0$.

18. Solve Laplace's equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

on the infinite rectangle $0 \le y \le L$, $0 \le x$ (see Figure 6.4) with the boundary values

$$u(x, 0) = 0 = u(x, L)$$
$$u(0, y) = f(y)$$
$$\frac{\partial u}{\partial x}(0, y) = g(y)$$

Show that the assumption that u is bounded implies that the third condition is unnecessary: the solution is uniquely determined by its boundary values.

19. Find the bounded solution of the differential equation $\Delta u + u = 0$ in the infinite rectangle (Figure 6.5) with the boundary conditions

$$u(x, 0) = 0 = u(x, L)$$
$$u(0, y) = f(y)$$

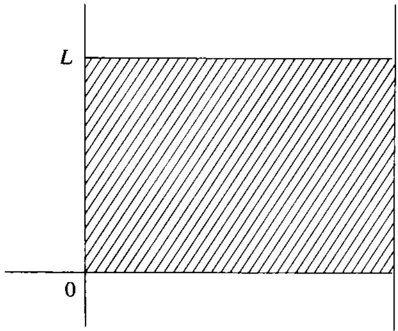

Figure 6.5

6.5 The Geometry of Fourier Expansions

We now return to the study of functions on the circle; that is, periodic functions of period 2π. We still have not studied the sense in which the Fourier series of a function converges to that function; we have only Corollaries 1 and 3 of Theorem 6.1 which deal with pointwise uniform convergence. Let us consider the real Fourier series of a continuous real-valued function f:

$$A_0 + \sum_{n=1}^{\infty} [A_n \cos nx + B_n \sin nx] \tag{6.47}$$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Since the Fourier series of a trigonometric function is itself, we find, by applying these definitions to $\cos nx$, $\sin nx$, that

$$\int_{-\pi}^{\pi} \cos nx \sin mx\,dx = 0 \qquad \text{all } n, m \tag{6.48}$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx\,dx = \begin{cases} 0 & n \neq m \\ \pi & n = m \neq 0 \\ 2\pi & n = m = 0 \end{cases} \tag{6.49}$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx\,dx = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases} \tag{6.50}$$

There is a geometric way of interpreting these equations which sheds light on the subject. We consider $C(\Gamma)$ as a vector space endowed with the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$$

This inner product, of course, defines a notion of distance (recall Section 1.11)

$$\|f - g\|_2 = \left[\int_{-\pi}^{\pi} |f(x) - g(x)|^2\,dx\right]^{1/2} \tag{6.51}$$

which is quite distinct from the uniform, or supremum distance

$$\|f - g\| = \max\{|f(x) - g(x)| : -\pi \le x \le \pi\}$$

We shall call the distance (6.51) the **mean square distance**, and we shall speak of convergence in this sense as **mean square convergence**. More precisely, $f_n \to f$ (mean square) if $\|f_n - f\|_2 \to 0$ as $n \to \infty$.

Now the importance of the equations above is that they imply that the functions $\cos nx$, $\sin nx$ are mutually orthogonal in the vector space $C(\Gamma)$ with this inner product. Thus, we can interpret (6.47) as an orthogonal expansion. Let us make these new definitions:

$$C_0(x) = \frac{1}{(2\pi)^{1/2}} \qquad C_n(x) = \frac{\cos nx}{\sqrt{\pi}} \qquad S_n(x) = \frac{\sin nx}{\sqrt{\pi}}$$

Then the collection C_n, S_n is, according to Equations (6.48)–(6.50), an orthonormal set. If f is any function on the circle,

$$A_0 = \frac{1}{(2\pi)^{1/2}} \int_{-\pi}^{\pi} f(x) \frac{1}{(2\pi)^{1/2}}\,dx = \frac{\langle f, C_0\rangle}{(2\pi)^{1/2}}$$

$$A_n = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \frac{\cos nx}{\sqrt{\pi}}\,dx = \frac{\langle f, C_n\rangle}{\sqrt{\pi}}$$

$$B_n = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \frac{\sin nx}{\sqrt{\pi}}\,dx = \frac{\langle f, S_n\rangle}{\sqrt{\pi}}$$

so the Fourier expansion (6.47) can be rewritten as

$$\langle f, C_0\rangle C_0 + \sum_{n=1}^{\infty} [\langle f, C_n\rangle C_n + \langle f, S_n\rangle S_n]$$

and is thus the infinite-dimensional analog of the orthogonal expansion of an element in an inner product space in terms of an orthonormal basis. This interpretation has important consequences for us.

Theorem 6.3. *Let f be a continuous function on the unit circle, and let* (6.47) *be its Fourier series.*

(i) *Among all trigonometric polynomials of degree at most N, the closest to f is*

$$A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx) \tag{6.52}$$

(ii) (*Bessel's inequality*)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx \geq A_0{}^2 + \frac{1}{2}\sum_{n=1}^{\infty} (A_n{}^2 + B_n{}^2) \tag{6.53}$$

Proof. In order to verify these facts, we use the basic theorem on orthogonal expansions (Theorem 1.8). The functions $C_0, C_1, \ldots, C_N, S_1, \ldots, S_N$ form an orthonormal basis for the space $\mathbf{S}_N$ of trigonometric polynomials of degree at most N. The orthogonal projection of f into this space is

$$f_0 = \langle f, C_0\rangle C_0 + \sum_{n=1}^{N} \langle f, C_n\rangle C_n + \langle f, S_n\rangle S_n$$

which is the same as (6.52). Thus, according to Theorem 1.8

(i) $\|f\|_2{}^2 = \|f_0\|_2{}^2 + \|f - f_0\|_2{}^2$

(ii) for any $w \in \mathbf{S}_N$, $\|f - f_0\|_2{}^2 \leq \|f - w\|_2{}^2$

(ii) directly implies Theorem 6.3(i). According to (i),

$$\|f\|_2{}^2 \geq \|f_0\|_2{}^2 = (\langle f, C_0\rangle)^2 + \sum_{n=1}^{N} (\langle f, C_n\rangle)^2 + (\langle f, S_n\rangle)^2$$

$$\|f\|_2{}^2 \geq 2\pi A_0{}^2 + \pi \sum_{n=1}^{N} A_n{}^2 + B_n{}^2$$

Since this is true for all N, we can take the limit on the right as $N \to \infty$, thus obtaining Bessel's inequality.

Now, it is clear that for trigonometric polynomials, Bessel's inequality is actually equality. For if f is such a trigonometric polynomial, there is an N such that $f \in \mathbf{S}_N$, so $f = f_0$. Thus, by (i) above $\|f\|^2 = \|f_0\|^2$, and $\|f_0\|^2$ is just the right-hand side of Bessel's inequality. Since any function can be uniformly approximated by trigonometric polynomials (although not necessarily by its Fourier series), we should expect Bessel's inequality to be always equality. This is the case.

Corollary. (Parseval's Equality) *If f is a continuous function on the unit circle and has the Fourier series* (6.47), *then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx = A_0{}^2 + \frac{1}{2}\sum_{n=1}^{\infty}(A_n{}^2 + B_n{}^2)$$

Proof. We continue the notation of Theorem 6.3. Let $\varepsilon > 0$ be given. By Corollary 4 of Theorem 6.1, there is a trigonometric polynomial w such that $\|w - f\| < \varepsilon$. Then

$$\|w - f\|_2{}^2 = \int_{-\pi}^{\pi} |w - f|^2\, dx \leq \|w - f\|^2 \int_{-\pi}^{\pi} dx < 2\pi\varepsilon^2$$

Now, since w is a trigonometric polynomial, there is an N such that $w \in \mathbf{S}_N$. Let f_0 be the projection of f into $\mathbf{S}_N$. Then by (i) and (ii),

$$\|f\|_2{}^2 = \|f_0\|_2{}^2 + \|f - f_0\|_2{}^2 \leq \|f_0\|_2{}^2 + \|f - w\|_2{}^2 \leq \|f_0\|_2{}^2 + 2\pi\varepsilon^2$$

This becomes, as in the above argument,

$$\int_{-\pi}^{\pi} |f(x)|^2\, dx \leq 2\pi A_0{}^2 + \pi \sum_{n=1}^{N} A_n{}^2 + B_n{}^2 + 2\pi\varepsilon^2$$

Since the sum to infinity only increases the right-hand side,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx \leq A_0{}^2 + \frac{1}{2}\sum_{n=1}^{N} (A_n{}^2 + B_n{}^2) + \varepsilon^2$$

Now, since ε was arbitrary we may let it tend to zero. The resulting inequality, together with Bessel's inequality, gives Parseval's equality.

Finally, we note that Parseval's equality can be expressed in terms of the expansion into a series of complex exponentials: $\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta}$. Since

$$\hat{f}(0) = A_0 \qquad \hat{f}(n) = \tfrac{1}{2}(A_n + iB_n) \qquad \hat{f}(-n) = \tfrac{1}{2}(A_n - iB_n)$$
$$A_0{}^2 = |\hat{f}(0)|^2 \qquad A_n{}^2 + B_n{}^2 = 2(|\hat{f}(n)|^2 + |\hat{f}(-n)|^2)$$

so we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2\, d\theta = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$$

Examples

16. Since $\cos^2\theta = \frac{1}{4}e^{-i2\theta} + \frac{1}{2} + \frac{1}{4}e^{i2\theta}$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^4\theta\, d\theta = \frac{1}{16} + \frac{1}{4} + \frac{1}{16} = \frac{3}{8}$$

17. $\pi^2 - \theta^2 = 2\pi^2/3 + 2\sum_{n \neq 0}^{\infty} (-1)^n e^{in\theta}/n^2$

$$\int_{-\pi}^{\pi} (\pi^2 - \theta^2)^2\, d\theta = \frac{16\pi^5}{15} = 2\pi\left[\frac{4\pi^4}{9} + 4\sum_{n\neq 0}\frac{1}{n^4}\right]$$

We conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

The partial sum to degree 3 of the Fourier series of $\pi^2 - \theta^2$ is

$$F_3(\theta) = \frac{2\pi^2}{3} - 2\cos\theta + \frac{1}{2}\cos 2\theta - \frac{2}{9}\cos 3\theta$$

The square of the mean square distance between $\theta^2 - \pi^2$ and this sum is

$$\sum_{n=4}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} - 1 - \frac{1}{16} - \frac{1}{81} \leq \frac{10}{9} - 1 - \frac{1}{16} - \frac{1}{81}$$
$$\leq \frac{8}{81} - \frac{1}{16} \leq \frac{3}{80}$$

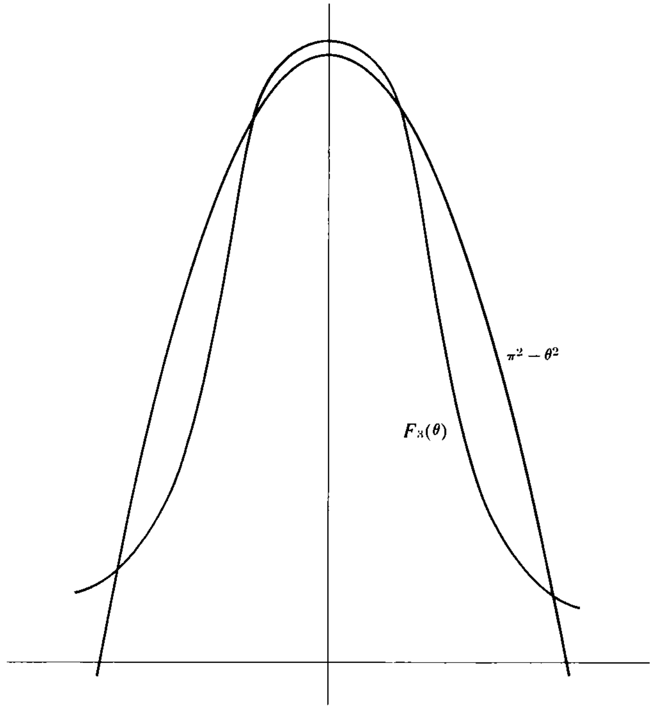

Figure 6.6

In Figure 6.6 the graphs of $\pi^2 - \theta^2$ and $F_3(\theta)$ are drawn.

18. $|\theta|$ has the Fourier expansion

$$\frac{\pi}{2} - \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{e^{i(2n+1)\theta}}{(2n+1)^2}$$

From Parseval's equality, we find

$$\frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{4}{\pi^2} 2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

The third partial sum of the Fourier series of $|\theta|$ is

$$F_3(\theta) = \frac{\pi}{2} - \frac{2}{\pi}\left(\cos\theta + \frac{\cos 3\theta}{9}\right)$$

The mean square distance between $|\theta|$ and this trigonometric polynomial is

$$\sum_{n=2}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96} - 1 - \frac{1}{81} \leq \frac{4}{96} - \frac{1}{81} \leq \frac{3}{96}$$

(see Figure 6.7).

Mean square approximation is interesting from the physical point of view. Consider the solution of the wave equation (suitably normalized)

$$u(x, t) = \sum_{n=0}^{\infty} (A_n \cos nt + B_n \sin nt) \sin nx \tag{6.54}$$

The (kinetic) energy of the wave at time t is proportional to

$$\int \left| \frac{\partial u}{\partial t} \right|^2 dx$$

Now, by Parseval's equality that can easily be computed in terms of the Fourier sine coefficients of $\partial u/\partial t$:

$$\frac{\partial u}{\partial t} = \sum_{n=0}^{\infty} n(B_n \cos nt - A_n \sin nt) \sin nx$$

$$\text{const} \int \left| \frac{\partial u}{\partial t} \right|^2 dx = (\Sigma n^2 (B_n \cos nt - A_n \sin nt)^2)$$

(Because of our normalizations, the constant is not relevant; it might as well

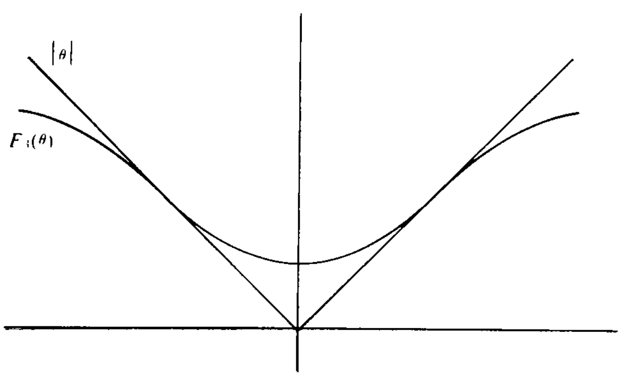

Figure 6.7

be 1.) Now the maximum value of the right-hand side is

$$\sum_{n=1}^{\infty} n^2(A_n^{\ 2} + B_n^{\ 2})$$

(see Problem 20) so this is the maximum kinetic energy of the wave. Now, according to our geometric considerations above, the Nth partial sum of (6.54) provides the best approximation to the solution wave in the sense of energy. Furthermore, the difference in energy levels between the solution wave and this approximation is readily computable, it is

$$\sum_{n>N} n^2(A_n^{\ 2} + B_n^{\ 2})$$

Since energy is the important concept in the study of waves, this mean square approximation is well suited for this study.

• EXERCISES

17. Compute these integrals by Fourier methods:

(a) $\displaystyle\int_{-\pi}^{\pi} \cos^8 3\theta \, d\theta$

(b) $\displaystyle\int_{-\pi}^{\pi} \sin^2 \mu\theta \, d\theta \qquad \mu$ not an integer

(c) $\displaystyle\int_{-\pi}^{\pi} \left[\frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \phi)}\right]^2 d\phi$

(d) $\displaystyle\int_{-\pi}^{\pi} \left[\cos\phi \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \phi)}\right]^2 d\phi$

(e) $\displaystyle\int_{-\pi}^{\pi} \theta^2 \, d\theta$

(f) $\displaystyle\int_{-\pi}^{\pi} \theta^4 \, d\theta$

18. Approximate the given function by a trigonometric polynomial to within 10^{-3} in mean.

(a) $|\theta|\,\theta$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\theta}{(n!)^{1/2}}$

(c) $\sin^3 \theta \cos \theta$

(d) $e^{\cos\theta}$

- **PROBLEMS**

20. Show that the maximum of $(B \cos nt - A \sin nt)^2$ is $B^2 + A^2$.

21. Let $\{f_n\}$ be a sequence of continuous functions on the circle. Show that if $f_n \to f$ uniformly, then $f_n \to f$ in mean. Show by example that the converse statement is false.

22. Prove: if f, g are integrable real-valued functions on the circle

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)}\,d\theta = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}$$

6.6 Differential Equations on the Circle

We now turn to a slightly different problem involving ordinary differential equations. We propose to find all **periodic** solutions of a linear constant coefficient equation. The particular theory which results is not in itself of vital importance, but it is worthwhile to study because of the symmetry of the results and because it presents the simplest example of the general theory of differential operators on compact manifolds.

As we have already seen, it is valuable in the theory of ordinary differential equations to allow complex-valued functions. We return then to our original form of the Fourier expansion of a function f: $\sum \hat{f}(n)e^{in\theta}$. Our first result concerns the computation of the Fourier coefficients of the derivative of a function.

Proposition 5. *Let f be a continuously differentiable function on the circle. The Fourier series of f' is obtainable by term by term differentiation. That is,*

$$\hat{f'}(n) = in\hat{f}(n) \tag{6.55}$$

Proof. The proof is by integration by parts.

$$\hat{f'}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(\theta)e^{in\theta}\,d\theta = \frac{1}{2\pi} f(\theta)e^{-in\theta}\Big|_{-\pi}^{\pi} + \frac{in}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta$$

$$= in\hat{f}(n)$$

Thus, if the differentiable function f has the Fourier series $\sum A_n e^{in\theta}$, then the Fourier series of f' is $\sum inA_n e^{in\theta}$. It follows from the fundamental theorem of calculus that we can also integrate Fourier series term by term, so long as it has no constant term: if f has the Fourier series $\sum A_n e^{in\theta}$, then

$\int_0^? f$ has the Fourier series $\sum (in)^{-1} A_n e^{in\theta}$. A useful consequence of Proposition 5. in conjunction with Bessel's inequality is that a continuously differentiable function is the sum of its Fourier series.

Proposition 6. *If f is a continuously differentiable function, then $f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{in\theta}$ holds for all θ.*

Proof. By Bessel's inequality

$$\sum_{n=-\infty}^{\infty} |\hat{f'}(n)|^2 < \infty$$

Using the above proposition we then obtain by Schwarz's inequality

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| = |\hat{f}(0)| + \sum_{n \neq 0}^{\infty} \left|\frac{\hat{f'}(n)}{in}\right| = |\hat{f}(0)| + \sum_{n \neq 0} \frac{1}{n} \left|\hat{f'}(n)\right|$$

$$\leq |\hat{f}(0)| + \left(\sum_{n \neq 0}^{\infty} \frac{1}{n^2}\right)^{1/2} \left(\sum_{n \neq 0} |\hat{f'}(n)|^2\right)^{1/2} < \infty$$

Thus $\sum |\hat{f}(n)| < \infty$, so Corollary 1 of Theorem 6.1 applies.

Now, suppose g is a continuous function on the circle. Given a polynomial $F(X) = X^k + \sum_{i=0}^{k-1} a_i X^i$, we want to find a periodic function f such that

$$f^{(k)} + \sum_{i=0}^{k-1} a_i f^{(i)} = g \tag{6.56}$$

The fact that we are interested in periodic functions is a new twist and the local results, such as Picard's theorem, are hardly applicable. For example, consider the simplest differential equation:

$$f' = g \tag{6.57}$$

By local considerations we know that f must be

$$f(\theta) = \int_{-\pi}^{\theta} \hat{g}(\phi)\, d\phi + c$$

However, f will be a periodic function only if $f(\pi) = f(-\pi) = c$: for this we must have $\int_{-\pi}^{\pi} g(\phi)\, d\phi = 0$. Thus (6.57) has a solution if and only if $\hat{g}(0) = 0$. We have already recognized this condition in the above discussion of integration of Fourier series. For by (6.55), if $f' = g$ we must have $in\hat{f}(n) = \hat{g}(n)$ for all n. This necessary condition shows up again by taking $n = 0$: we must have $\hat{g}(0) = 0$.

Now we return to the general case (6.56). If we look at the Fourier series of both sides this becomes $F(in)\hat{f}(n) = \hat{g}(n)$. Thus we must have $\hat{g}(n) = 0$ whenever $F(in) = 0$. Otherwise, the equation does not have a solution. On the other hand, if this condition is satisfied, then the equation is easily solved since we must have $\hat{f}(n) = F(in)^{-1}\hat{g}(n)$. The solution is the function whose Fourier series is

$$\sum_{n=-\infty}^{\infty} \frac{\hat{g}(n)}{F(in)} e^{in\theta} \tag{6.58}$$

Theorem 6.4. *Let $F(X) = \sum_{i=0}^{k-1} c_i X^i$, and let $n_1, \ldots, n_\sigma$ be the roots of $F(in) = 0$. Let L_F be the differential operator*

$$L_F(f) = \sum_{i=0}^{k} c_i f^{(i)}$$

(i) *The space of periodic solutions of $L_F(f) = 0$ is spanned by* $\exp(in_1\theta), \ldots, \exp(in_\sigma\theta)$.

(ii) *Given any periodic function g, the equation $L_F f = g$ has a solution if and only if $\hat{g}(n_i) = 0$, $1 \le i \le \sigma$. The solution is uniquely determined by specifying the Fourier coefficients $\hat{f}(n_i)$, $1 \le i \le \sigma$.*

Proof. The Fourier coefficients of $L_F(f)$ are $\{F(in)\hat{f}(n)\}$. Now if $L_F(f) = 0$, we must have $F(in)\hat{f}(n) = 0$ for all n, so $\hat{f}(n) = 0$ necessarily except when $F(in) = 0$. Since $n_1, \ldots, n_\sigma$ are the roots of this equation, (i) is proven.

If g is a periodic function and $L_F(f) = g$, we must have $F(in)\hat{f}(n) = \hat{g}(n)$. Thus $\hat{g}(n_i) = 0$, $1 \le i \le \sigma$ is a necessary condition for this equation. Suppose now that this condition is satisfied. Then if f is a solution we must have

$$\hat{f}(n) = \frac{\hat{g}(n)}{F(in)} \qquad n \ne n_1, \ldots, n_\sigma \tag{6.59}$$

and the $\hat{f}(n_i)$, $1 \le i \le \sigma$ can be freely chosen. Upon specification of these coefficients the Fourier series of f is uniquely determined. The only question is this: are the numbers (6.59) the Fourier coefficients of a function? The answer is *yes* when F is of degree at least one. For then $|F(in)| \ge C\,|n|$ for some constant C and all sufficiently large n (Problem 24), and thus

$$\sum \left|\frac{\hat{g}(n)}{F(in)}\right| \le \sum C \left|\frac{\hat{g}(n)}{n}\right| \le C \left(\sum \frac{1}{n^2}\right)^{1/2} (\sum |\hat{g}(n)^1|^2)^{1/2} < \infty \tag{6.60}$$

for the tail end of the series, and thus the sum of the whole series is finite. Hence the Fourier series

$$f(\theta) = \sum_{n=-\infty}^{\infty} \frac{\hat{g}(n)}{F(in)} e^{in\theta} \tag{6.61}$$

converges uniformly to a continuous function. The theorem is thus proven.

We can get a much better looking form for the solution, if the degree of F is large enough (at least second degree). For then

$$\tilde{F}(\theta) = \sum_{\substack{n=-\infty \\ n \neq n_1, \cdots, n_\sigma}}^{\infty} \frac{e^{in\theta}}{F(in)} \tag{6.62}$$

defines a continuous function (Problem 24) and the solution (6.61) is given by

$$\begin{aligned} f(\theta) &= \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{F(in)} \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\phi) e^{in\phi}\, d\phi \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\phi) \sum_{n=-\infty}^{\infty} \frac{e^{in(\theta-\phi)}}{F(in)}\, d\phi \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\phi) \tilde{F}(\theta-\phi)\, d\phi \end{aligned}$$

using (6.61). We can now write the conclusions of Theorem 6.4 explicitly in terms of an integral formula.

Theorem 6.5. *Let $F(X) = \sum_{i=0}^{k} c_i X^i$ $(k \geq 2)$, and let $n_1, \ldots, n_\sigma$ be the solutions of $F(in) = 0$. Let L_F be the differential operator defined by the polynomial F. Let*

$$\tilde{F}(\theta) = \sum_{\substack{n=-\infty \\ n \neq n_1, \cdots, n_\sigma}}^{\infty} \frac{e^{in\theta}}{F(in)}$$

Then the equation $L_F(f) = g$ has a solution if and only if $\hat{g}(n_i) = 0$,$1 \leq i \leq \sigma$. All solutions are of the form

$$f(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\phi) \tilde{F}(\theta-\phi)\, d\phi + \sum_{j=1}^{\sigma} c_j \exp(in_j\theta) \tag{6.63}$$

Thus a constant coefficient differential operator on the circle has an inverse of the form (6.63) (defined on its range), called an **integral operator**.

Examples

19. Find a periodic solution of $f'' - f = \cos 2\theta$. Now

$$g(\theta) = \cos 2\theta = \tfrac{1}{2}(e^{i2\theta} + e^{-i2\theta}).$$

The characteristic polynomial is $F(X) = X^2 - 1$ and $F(in) = -n^2 - 1$ has no roots. Thus there exists a unique solution and it is given by (6.58):

$$\begin{aligned} f(\theta) &= \sum_{n=-\infty}^{\infty} \frac{\hat{g}(n)e^{in\theta}}{-n^2 - 1} = -\frac{1}{2}\left(\frac{e^{i2\theta}}{5} + \frac{e^{-i2\theta}}{5}\right) \\ &= -\frac{1}{5}\cos 2\theta \end{aligned}$$

20. Solve: $f'' - 3f' + 2f = \pi^2 - \theta^2$. The characteristic polynomial is $F(X) = (X^2 - 3X + 2) = (X - 1)(X - 2)$ so that again $F(in) = 0$ has no integral roots. Since $\pi^2 - \theta^2$ has (by Example 4) the Fourier series

$$\frac{2\pi^2}{3} + 2\sum_{n\neq 0}(-1)^n \frac{e^{in\theta}}{n^2}$$

the solution is (by 6.58)

$$f(\theta) = \frac{\pi^2}{3} - 2\sum_{n\neq 0}(-1)^n \frac{e^{in\theta}}{n^2(n^2 + 3in - 2)}$$

21. $f^{(k)} = g$. This has a solution if $\hat{g}(0) = 0$. In this case the solution is given by

$$f(\theta) = C + \sum_{n\neq 0} \hat{g}(n) \frac{e^{in\theta}}{(in)^k}$$

22. Find all solutions of $f'' + 4f = 0$. Here, the roots of $X^2 + 4 = 0$ are $\pm 2i$, therefore, all solutions are periodic of period 2π: $e^{2i\theta}$, $e^{-2i\theta}$ span the space of solutions. Notice, however, that there are no solutions of $f'' + 5f = 0$ which are periodic.

• EXERCISES

19. Find all periodic solutions of these differential equations:
    (a) $y'' + 2iy' + 15y = 0$
    (b) $y^{(5)} - y^{(4)} + 10y^{(3)} - 10y'' + 9y' - 9y = 0$
    (c) $y^{(4)} + 2y'' + 1 = 0$
20. Find periodic solutions of these differential equations:
    (a) $y^{(4)} + 2y'' + y = \sin 5\theta + \cos 5\theta$
    (b) $y'' + 6y' + 9y = \pi^2 - \theta^2$
    (c) $y^{(5)} + y = \exp(\cos\theta)$

• PROBLEMS

23. Suppose F is a polynomial of degree at least k. Show that there is a $C > 0$, and an integer N such that $|F(in)| \geq C\,|n|^k$ for $n \geq N$.

24. Show that if F is a polynomial of degree at least 2, then

$$\tilde{F}(\theta) = \sum_{\substack{n=-\infty \\ n \text{ not a root}}}^{\infty} \frac{e^{in\theta}}{F(in)}$$

is a continuous function on the circle.

25. If f, g are two continuous functions on the circle, define $f * g$, the **convolution** of f and g, by

$$(f * g)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)g(\theta - \phi)\,d\phi$$

(a) Show that $f * g = g * f$.
(b) Show that the Fourier coefficients of $f * g$ are $\hat{f}(n)\hat{g}(n)$.
(c) Show that the differential equation $L_F(f) = g$, where L_F is the constant coefficient operator associated to the polynomial F is solved by $f = g * \tilde{F}$, where $\tilde{F}$ has the Fourier coefficients $F(in)^{-1}$.

26. Let

$$P_1(r, t) = \int_{-\pi}^{t} P(r, \tau)\,d\tau$$

(a) Show that $\lim_{r\to 1} P_1(r, t)$ is 0 if $t < 0$, and 1 if $t > 0$.
(b) Show that for any C^1 function g on the circle

$$g(\theta) = \lim_{r\to 1}\int_{-\pi}^{\pi} g'(\phi)P_1(r, \theta - \phi)\,d\phi$$

(*Hint:* $\lim_{r\to 1} P_1(r, \theta)$ "has the Fourier series" $\sum_{n=-\infty}^{\infty} e^{in\theta}/in$.)

6.7 Taylor Series and Fourier Series

If we now take the attitude that the unit circle is the boundary of the unit disk we discover connections between Fourier series and Taylor series which are of enormous significance in complex function theory. These connections cannot be fully exploited until we learn the fact (in the next chapter) that complex differentiable functions can be expanded in a power series. In this section we shall explore the relationship between the Fourier and Taylor expansions of such a function defined on the unit disk, assuming its Taylor series converges on the disk.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the unit disk. In polar coordinates this becomes

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \tag{6.64}$$

which is, for each r, a Fourier series. Using the Fourier theoretic material at hand we get a most remarkable collection of integral formulas for functions which are sums of convergent power series.

Theorem 6.6. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a convergent power series in $\{|z| \leq 1\}$. Then we have these equations.*

(i) *For each $r \leq 1$,*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(re^{i\theta})e^{-in\theta}\,d\theta = a_n = \frac{1}{n!}f^{(n)}(0) \qquad n \geq 0$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(re^{i\theta})e^{in\theta}\,d\theta = 0 \qquad n > 0$$

(ii) $$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}\,d\phi$$

(iii) $$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\frac{e^{i\phi}}{e^{i\phi}-z}\,d\phi$$

for $z = re^{i\theta}$, $r < 1$.

Proof. By Equation (6.64), for fixed r, $a_n r^n$ is the nth Fourier coefficient of $f(re^{i\theta})$ for $n \geq 0$, and for negative n the Fourier coefficient vanishes. This is just what part (i) says explicitly. Equation (6.64) also says that $f(re^{i\theta})$ is the Poisson integral of $f(e^{i\theta})$ and thus we obtain (ii). Then (iii) follows from resumming the series, using the fact that the negative coefficients vanish. Explicitly, we have

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{z^n}{2\pi} \int_{-\pi}^{\pi} f(\phi) e^{-in\phi}\, d\phi \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) \sum_{n=0}^{\infty} \left(\frac{z}{e^{i\phi}}\right)^n d\phi \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) \frac{e^{i\phi}}{e^{i\phi} - z}\, d\phi \end{aligned}$$

the last change being accomplished by summing the geometric series.

There are several more or less immediate conclusions one can draw from the above theorem. First of all, the sum of a convergent power series on the unit disk is completely determined by its boundary values, by Equation (iii), known as Cauchy's formula. This of course follows from the maximum principle verified in the last chapter for analytic functions. The Cauchy formula itself implies the maximum principle (see Problem 27). A more important implication is that the sum of a convergent power series in the disk is analytic; that is, it can be expanded in a power series centered at any point.

Corollary. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in the disk $\{|z| \le 1\}$. Then for any z_0, $|z_0| < 1$, f can be expanded in a power series centered at z_0.*

Proof. By Cauchy's formula

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\phi}) \frac{e^{i\phi}}{e^{i\phi} - z}\, d\phi$$

Now

$$\frac{1}{e^{i\phi} - z} = \frac{1}{e^{i\phi} - z_0 - (z - z_0)} = \frac{1}{e^{i\phi} - z_0} \cdot \left(1 - \frac{z - z_0}{e^{i\phi} - z_0}\right)^{-1}$$

In the disk $\{z \in C\colon |z - z_0| < 1 - |z_0|$, the last factor is the sum of a geometric series:

$$\left(1 - \frac{z - z_0}{e^{i\phi} - z_0}\right)^{-1} = \sum_{n=0}^{\infty} \left(\frac{z - z_0}{e^{i\phi} - z_0}\right)^n$$

which we may substitute in the integral. We obtain

$$f(z) = \sum_{n=0}^{\infty} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\phi}) \frac{e^{i\phi}}{(e^{i\phi} - z_0)^{n+1}} \, d\phi \right] (z - z_0)^n$$

the series being convergent for all z such that $|z - z_0| < 1 - |z_0|$. As a consequence we have still more integral formulas:

$$f^{(n)}(z_0) = \frac{n!}{2\pi} \int_{-\pi}^{\pi} f(e^{i\phi}) \frac{e^{i\phi}}{(e^{i\phi} - z_0)^{n+1}} \, d\phi$$

for any z_0, $|z_0| < 1$.

We shall see in the next chapter that these integral formulas can be explained in yet another way (basically the fundamental theorem of calculus) and are just very special cases of general formulas. We conclude now with an approximation theorem which should be contrasted with the Weierstrass approximation theorem (Problem 7) for a real variable.

Theorem 6.7. *Let f be a continuous function on the circle. f is approximable by polynomials in z if and only if*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{in\theta} \, d\theta = 0 \qquad n > 0 \tag{6.65}$$

Proof. If f is approximable by polynomials, there is a sequence $\{f_k\}$ of polynomials such that $f_k \to f$ uniformly. Since (6.65) is readily verified for any polynomial, it thus holds also for f, by continuity of the integral.

Conversely, if (6.65) is verified, then the Poisson transform of f is the sum of a convergent power series:

$$Pf(z) = \sum_{n=0}^{\infty} a_n z^n \qquad |z| < 1 \tag{6.66}$$

Since $Pf(re^{i\theta}) \to f(\theta)$ as $r \to 1$, then given $\varepsilon > 0$, there is an $r < 1$ such that $|Pf(re^{i\theta}) - f(\theta)| < \varepsilon$ for all θ.

Now on the circle $|z| = r$, the series (6.66) converges uniformly, so there is an N such that

$$\left| Pf(z) - \sum_{n=0}^{N} a_n z^n \right| < \varepsilon \qquad |z| = r$$

Then, on the unit circle,

$$\left|f(z) - \sum_{n=0}^{N} a_n r^n z^n\right| = \left|f(e^{i\theta}) - \sum_{n=0}^{N} a^n r^n e^{in\theta}\right|$$

$$\leq |f(e^{i\theta}) - Pf(re^{i\theta})| + \left|Pf(re^{i\theta}) - \sum_{n=0}^{N} a_n(re^{i\theta})^n\right| < 2\varepsilon$$

independently of θ.

● **EXERCISES**

21. Integrate:

(a) $\displaystyle\int_{-\pi}^{\pi} \frac{e^{3i\theta} + 4e^{2i\theta} + e^{i\theta}}{2e^{i\theta} - 1}\, d\theta$

(b) $\displaystyle\int_{-\pi}^{\pi} \frac{e^{ik\theta}}{(e^{i\theta} + 1/4)^n}\, d\theta$ $\qquad k, n$ positive integers

● **PROBLEMS**

27. Deduce the maximum principle for convergent power series on the disk from Cauchy's formula.

28. Using the results of Problem 11, verify that these assertions for a function defined on the disk are equivalent:

(a) $f(z) = \sum_{n=1}^{\infty} a_n z^n$

(b) f is complex differentiable.

(c) f is uniformly approximable by polynomials.

(d) f is harmonic and $\hat{f}(-n) = 0$ for $n > 0$.

6.8 Summary

The function $f(t) = \exp(2\pi it/L)$ wraps the real line around the circle so that every interval of length L covers the circle once. The collection of periodic functions of period L may be viewed as the collection of functions on the circle.

If f is a piecewise continuous function on the circle, its nth *Fourier coefficient* is

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)e^{in\phi}\, d\phi$$

The *Fourier series* of f is the series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{i\theta}$$

The Poisson transform of f is the function on the unit disk given by

$$Pf(r, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}\,d\phi = \sum_{n=-\infty}^{\infty} \hat{f}(n)r^{(n)}e^{in\theta}$$

Theorem. *If f is a continuous function of the circle and g is the function on the disk defined by*

$$g(r, \theta) = Pf(r, \theta) \qquad r < 1$$
$$g(1, \theta) = f(\theta)$$

then g is continuous on the disk and **harmonic** *(satisfies Laplace's equation) inside*:

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0 \qquad \text{for } r < 1$$

Theorem. *If f is continuously differentiable on the circle, then it is the sum of its Fourier series*:

$$f(\theta) = \sum_{v=-\infty}^{\infty} \hat{f}(n)e^{in\theta}$$

Suppose u is harmonic in a closed and bounded domain D in the plane. Then

(i) if $\Delta(a, R) \subset D$

$$u(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(a + Re^{i\theta})\,d\theta \qquad \textit{(mean value property)}$$

(ii) if $u < M$ on ∂D, $u < M$ inside D (*maximum principle*)

A function harmonic on a closed and bounded domain is uniquely determined by its boundary values.

If the real-valued function f has the Fourier series $\sum C_n e^{in\theta}$, we can rewrite it as

$$A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + B_n \sin n\theta$$

where

$$A_0 = C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\, d\phi$$

$$A_n = 2\,\mathrm{Re}\, C_n = C_n + C_{-n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi) \cos n\phi\, d\phi$$

$$B_n = 2\,\mathrm{Im}\, C_n = -i(C_n - C_{-n}) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi) \sin n\phi\, d\phi$$

If f is a C^1 periodic function of period L, it can be expanded in a *Fourier cosine series*

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{\pi n x}{L}$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\, dx \qquad A_n = \frac{2}{L}\int_0^L f(x) \cos \frac{\pi n x}{L}\, dx$$

or a *Fourier sine series*

$$f(x) = \sum_{k=1}^{\infty} B_n \sin \frac{\pi n x}{L}$$

$$B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{\pi n x}{L}$$

THE WAVE EQUATION. Given the C^2 periodic functions f, g of period L the equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

with the boundary data

$$y(0, t) = 0 \qquad y(L, t) = 0$$

and the initial data

$$y(x, 0) = f(x) \qquad \frac{\partial y}{\partial t}(x, 0) = g(x)$$

has a solution. The solution is given by

$$y(x, t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{\pi n t}{Lc} + B_n \sin \frac{\pi n t}{Lc} \right] \sin \frac{\pi n x}{L}$$

where

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi n x}{L}\right) dx \qquad B_n = \frac{2c}{n} \int_0^L g(x) \sin\left(\frac{\pi n x}{L}\right) dx$$

THE HEAT EQUATION. Given the C^2 periodic function f of period L the equation

$$\frac{1}{k^2} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with the boundary data

$$u(0, t) = 0 = u(L, t)$$

and the initial condition

$$u(x, 0) = f(x)$$

has a solution given by

$$u(x, t) = \sum_{n=1}^{\infty} A_n \exp(-C^2 n^2 t) \sin \frac{\pi n x}{L}$$

where $C = \pi/Lk$ and

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{\pi n x}{L} dx$$

Consider $C(\Gamma)$ as endowed with the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x)\, dx$$

Let

$$C_0(x) = \frac{1}{(2\pi)^{1/2}} \qquad C_n(x) = \frac{\cos nx}{\sqrt{\pi}} \qquad S_n(x) = \frac{\sin nx}{\sqrt{\pi}}$$

$\{C_n, S_n\}$ is an orthonormal set. The Fourier series of a function f can be rewritten as

$$\sum_{n=0}^{\infty} \langle f, C_n \rangle C_n + \sum_{n=1}^{\infty} \langle f, S_n \rangle S_n$$

PARSEVAL'S EQUALITY

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = A_0{}^2 + \frac{1}{2} \sum_{n=1}^{\infty} (A_n{}^2 + B_n{}^2)$$

DIFFERENTIAL EQUATIONS ON THE CIRCLE. Let F be a polynomial and let $n_1, \ldots, n_\sigma$ be the integer solutions of $F(in) = 0$. Let L_F be the differential operator defined by the polynomial F. Let

$$\tilde{F}(\theta) = \sum_{\substack{n=-\infty \\ n \neq n_1, \ldots, n_\sigma}}^{\infty} \frac{e^{in\theta}}{F(in)}$$

Then the equation $L_F(f) = g$ has a solution if and only if $g(n_i) = 0$, $1 \leq i \leq \sigma$. All solutions are of the form

$$f(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\phi) \tilde{F}(\theta - \phi) \, d\phi + \sum_{j=1}^{\sigma} c_j \exp(in_j \theta)$$

Theorem. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a convergent power series on the unit disk. Then these equations are valid:*

(i) $$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{i\theta}) e^{-in\theta} \, d\theta = a_n = \frac{1}{n!} f^{(n)}(0) \qquad n \geq 0, r \leq 1$$

(ii) $$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{i\theta}) e^{in\theta} \, d\theta = 0, n > 0 \qquad r \leq 1$$

(iii) $$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\phi}) \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \phi)} \, d\phi$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\phi}) \frac{e^{i\phi}}{e^{i\phi} - z} \, d\phi$$

If f is complex differentiable in a domain D, it can be expanded in a power series in some disk centered at any point in D.

• FURTHER READING

The theory of Fourier series is exposed in these texts:

R. Seeley, *An Introduction to Fourier Series and Transforms*, W. A. Benjamin, Inc., New York, 1966.

Kreider, Kuller, Ostberg, Perkins, *An Introduction to Linear Analysis*, Addison-Wesley, Reading, Mass., 1966.

Hardy and Rogosinski, *Fourier Series*, Oxford University Press, New York, 1956.

Further applications to physics and the development of other partial differential equations can be pursued in

E. Butkov, *Mathematical Physics*, Addison-Wesley, Reading, Mass., 1968.

O. D. Kellogg, *Foundations of Potential Theory*, Dover, New York, 1953.

• MISCELLANEOUS PROBLEMS

29. Let

$$f(\theta) = \begin{cases} 1 & \theta \geq 0 \\ 0 & \theta < 0 \end{cases}$$

Show that $\sum_{n=-\infty}^{\infty} \hat{f}(n) = 1/2$ (see Example 7). What is

$$\sum_{n=-\infty}^{\infty} (-1)^n \hat{f}(n)?$$

30. Let f be a piecewise continuous function on the circle. Suppose f has a *jump discontinuity* at 0; that is, the limits

$$\lim_{\substack{x \to 0 \\ x < 0}} f(x) = a \qquad \lim_{\substack{x \to 0 \\ x > 0}} f(x) = b$$

both exist, but are different. Show that

$$\lim_{r \to 1} Pf(r, 0) = \frac{1}{2}(a + b)$$

(*Hint:* Follow the proof of Theorem 6.1 for $\theta \geq 0$, $\theta \leq 0$ independently, using the substitution

$$\frac{1}{2} = \int_0^{\pi} P(r, -\phi)\, d\phi = \int_{-\pi}^{0} P(r, -\phi)\, d\phi)$$

31. Show that f is an infinitely differentiable function on the circle if and only if this condition on the Fourier coefficients is satisfied: for every $k \geq 0$ there is an $M > 0$ such that

$$|\hat{f}(n)| \leq \frac{M}{|n|^k} \qquad \text{for all } n$$

32. Suppose

$$f(z) = \frac{P(z)}{Q(z)}$$

where P is analytic on $\{|z| \leq 1\}$ and Q is a polynomial. Show that there is an integer N and complex numbers $a_0, \ldots, a_N$ such that

$$a_N \hat{f}(k - N) + \cdots + a_0 \hat{f}(k) = 0 \qquad \text{for all } k < 0$$

(*Hint:* Let $Q(z) = a_N z^N + \cdots + a_0$.) State and prove the converse assertion.

33. Suppose that f is complex differentiable in the annulus $\{r \leq |z| \leq R\}$. Using the polar form of the Cauchy–Riemann equations, show that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

where the a_n can be computed from the Fourier coefficients $\hat{f}(n)$ of $f(\rho e^{i\theta})$ for any ρ between r and R.

34. Show that if f is analytic in the punctured disk $\{0 < |z| \leq R\}$ and bounded, then f extends analytically to the entire disk.

35. Show that if u is harmonic in the disk $\{|z - a| \leq R\}$, then for every $r < R$,

$$u(a + re^{i\theta}) = \frac{1}{2\pi R} \int_{-\pi}^{\pi} u(a + Re^{i\phi}) \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\theta - \phi)} \, d\phi$$

36. **(Harnack's principle)** If u is harmonic and nonnegative in the disk $\{|z - a| \leq R\}$, then

$$\frac{R - r}{R + r} \leq u(a + re^{i\theta}) \leq \frac{R + r}{R - r}$$

37. Suppose $\{u_n\}$ is a sequence of nonnegative harmonic functions on the disk $\{|z - a| \leq R\}$. Suppose also that $\sum u_n(a) < \infty$. Then

$$u(z) = \sum u_n(z)$$

converges for all z in that disk, and u is harmonic.

38. Show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$$

39. Verify the trigonometric identity

$$\frac{1}{2} + \sum_{n=1}^{N} \cos 2n\theta = \frac{\sin(2N+1)\theta}{2 \sin \theta}$$

(*Hint:* The sum to be evaluated is

$$\frac{1}{2} + 2 \, Re \sum_{n=1}^{N} (e^{2i\theta})^n)$$

40. Using the identity of Problem 39, obtain *Dirichlet's integral* for the partial sums of the Fourier series of f:

$$S_N(\theta) = A_0 + \sum_{n=1}^{N} (A_n \cos n\theta + B_n \sin n\theta)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(N+\frac{1}{2})\phi}{\sin \frac{1}{2}\phi} f(\theta+\phi) \, d\phi$$

$$= \frac{1}{2\pi} \int_{0}^{\pi} \frac{\sin(N+\frac{1}{2})\phi}{\sin \frac{1}{2}\phi} [f(\theta+\phi) + f(\theta-\phi)] \, d\phi$$

41. Using the Dirichlet integral (Problem 40), verify that for f a continuous function on the circle which is differentiable at the point θ_0, then

$$f(\theta_0) = A_0 + \sum_{n=1}^{\infty} (A_n \cos n\theta_0 + B_n \sin n\theta_0)$$

(*Hint:*

$$S_N(\theta_0) - f(\theta_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin\left(N + \frac{1}{2}\right) \phi \frac{f(\theta_0+\phi) - f(\theta_0)}{\sin \frac{1}{2}\phi} \, d\phi$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin N\phi \left[\cos \frac{\phi}{2} \frac{f(\theta_0+\phi) - f(\theta_0)}{\sin \frac{1}{2}\phi}\right] + \cos N\phi [f(\theta_0+\phi) - f(\theta_0)] \, d\phi$$

The expressions in brackets are continuous functions.)

42. Solve the differential equation

$$y^{(4)} = g$$
$$y(0) = 0 \qquad y(L) = 0$$

by Fourier methods, where (a) g is constant, (b) $g(x) = L - x$.

43. Suppose we want to study the problem of heat transfer through a homogeneous rod with insulated ends: heat does not flow through the ends. If the rod is assumed to lie on the interval $0 \le x \le L$ this amounts to the boundary conditions

$$\frac{\partial u}{\partial x}(0, t) = 0 = \frac{\partial u}{\partial x}(L, t)$$

The initial condition may be given either as the temperature distribution $(u(x, 0))$ or the initial heat flow $[(\partial u/\partial t)(x, 0)]$. Show that with these boundary conditions and either kind of initial condition the heat equation (6.43) can be uniquely solved.

44. Solve the insulated end heat problem with (a) constant initial temperature, (b) initial temperature $= \cos(x/L)$, (c) initial heat flow $= x(x - L)$.

45. Suppose that u is a real-valued function harmonic in the unit disk. Show that u is the real part of an analytic function. (*Hint:* Write $u(z) = a_0 + \sum_{n=1}^{\infty} (a_{-n} z^{-n} + a_n z^n)$, and add a pure imaginary-valued harmonic function with the same negative Fourier coefficients.)

46. If u is harmonic and real-valued on the unit disk, there is a unique harmonic real-valued function v such that $v(0) = 0$, $u + iv$ is analytic. Using the relation between the Fourier expansions of v and u (Problem 45) find an integral form for v in terms of the boundary values of u.

47. If u and v are as in Problem 45, show that the families of curves $\{u = \text{constant}\}$, $\{v = \text{constant}\}$ are orthogonal.

48. **(The convolution transform)** Let g be a continuous function on the circle and define the transformation $G\colon C(\Gamma) \to C(\Gamma)$ by

$$G(f)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) g(\theta - \phi)\, d\phi$$

Show that (a) the eigenvalues of G are the Fourier coefficients $\hat{g}(n)$ of g.

(b) The nonzero eigenvalues of g form a sequence converging to zero.

(c) The eigenspaces associated to the nonzero eigenvalues are finite dimensional.

(d) The Fourier series of $G(f)$ is

$$\sum \hat{g}(n) \hat{f}(n) e^{in\theta}$$

(e) $G(u) = f$ has a solution if and only if f is orthogonal to the kernel of G.

49. Under what conditions is the convolution transform (Problem 48) symmetric; skew-symmetric; one-to-one?

The Laplace Transform

50. The Laplace transform is useful in the study of differential equations defined on the positive real axis, R^+. If f is a bounded function on R^+, define

$$L(f)(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Show that $L(f)$ is defined for all $s > 0$.

51. A function f defined on R^+ is *of exponential order* s_0 if $\exp(-s_0 t) f(t)$ is a bounded function. Show that for such a function, $L(f)$ is defined for $s \geq s_0$.

52. Compute these Laplace transforms:

(a) $L(1)(s) = \dfrac{1}{s}$

(b) $L(t^n) = \dfrac{n!}{s^{n+1}}$

(c) $L(e^t) = \dfrac{1}{(s-1)}$

(d) $L(e^{-t}) = ?$

53. Verify these properties of the Laplace transform:

(a) L is linear.

(b) if $f_a(x) = \begin{cases} f(x-a) & x \geq a, \text{ for } a \geq 0 \\ 0 & x < 0 \end{cases}$

$$L(f_a) = e^{-as} L(f)$$

(c) $L(f') = sL(f) - f(0)$

54. Notice that by the above problems we see that the Laplace transform of a polynomial is a polynomial in $1/s$. More generally, we might expect at least that if f is of exponential type, $L(f)(s) \to 0$ as $s \to \infty$. Is this true?

55. Because of property (c) in Problem 53, Laplace transformation transforms a differential equation into an algebraic problem. For example, suppose we want to solve $y'' + y = 1$, $y(0) = 0$, $y'(0) = 0$ on R^+. If f is a

solution we must have

$$L(f') = sL(f) - f(0)$$
$$L(f'') = sL(f') - f'(0) = s^2L(f) - sf(0) - f'(0)$$

Thus, using the differential equation and the initial conditions

$$L(f'' + f) = L(1)$$

$$s^2L(f) + L(f) = \frac{1}{s}$$

$$L(f) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{1}{2}\left(\frac{1}{s+i}\right) - \frac{1}{2}\left(\frac{1}{s-i}\right)$$

Reading Problem 52(a)–(d) backward, we obtain the solution

$$f(t) = 1 - \tfrac{1}{2}(e^{it} + e^{-it}) = 1 - \cos t$$

Solve these equations by Laplace transformation:

(a) $y'' + y = 1$ $\quad y(0) = 0 \quad y'(0) = 1$
(b) $y'' - y = e^t$ $\quad y(0) = 1 \quad y'(0) = 0$
(c) $y' - 2y = e^{-t}$ $\quad y(0) = 1$
(d) $y''' + 3y' + 2y = e^{-t} + t$ $\quad y(0) = 1 \quad y'(0) = 0$

56. Solve these systems by Laplace transformation:

(a) $y_1' + y_2 = e^{-2t} \quad y_1(0) = 0 = y_2(0)$
$\quad y_2' + 2y_1 = 1.$

(b) $y_1'' - y_2' = y_1 \quad y_1(0) = 0 \quad y_1'(0) = 1$
$\quad y_2'' + y_1' = 2y_2 \quad y_2(0) = 1 \quad y_1'(0) = 0.$

57. Define this convolution for continuous functions on R^+

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

Show that $L(f * g) = L(f)L(g)$.

58. Solve the differential equation

$$y'' + 2y' + y = f(t) \qquad y(0) = 0 \qquad y'(0) = 1$$

where f is of exponential type (recall Problem 51).

59. Show that the function of a complex variable

$$L(f)(z) = \int_0^\infty e^{zt} f(t)\,dt$$

is complex differentiable (if f is continuous and bounded) in the domain $\{z \in C: \operatorname{Re} z > 0\}$.

The Fourier Transform

60. We now consider the collection of continuous functions defined on all of R. We shall discuss the Fourier transform, which is the analog of Fourier series:

f periodic	f defined on R
$\hat{f}(n) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta$	$\hat{f}(\xi) = \dfrac{1}{(2\pi)^{1/2}}\displaystyle\int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,d\xi$
Fourier series of $f = \displaystyle\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta}$	$F(f)(x) = \dfrac{1}{(2\pi)^{1/2}}\displaystyle\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi x}\,d\xi$

Of course, since we are working on an infinite interval, we want to be assured that the Fourier transform is defined. The most appropriate class of functions will come out of these observations:

(i) If f is integrable on R, $\hat{f}$ is a bounded continuous function.
(ii) The Fourier transform $f \to \hat{f}$ is linear.
(iii) $(f')^{\wedge} = (i\xi)\hat{f}$
(iv) $\hat{f}' = (ixf)^{\wedge}$

Since Fourier transformation interchanges the operations of differentiation and multiplication, we select the class of functions such that the effect of all such operations produces an integrable function. This is the **Schwartz class $S(R)$ of test functions**: f is a Schwartz function, $f \in S(R)$, if and only if f is C^∞, and for all positive integers n and k the function

$$\frac{d^k}{dx^k}(x^n f)$$

is integrable on R. Show that $f \in S(R)$ implies $\hat{f} \in S(R)$. (*Hint:* "$(x^n f)^{(k)}$ integrable" implies "$(i\xi)^{-k}\hat{f}^{(n)}$ bounded" implies "$\xi^{k-2}\hat{f}^{(n)}$ integrable.")

61. For $f, g \in S(R)$ define the convolution by

$$(f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy$$

Show that $(f * g)^{\wedge} = \hat{f}\hat{g}$.

62. Borrowing again from the theory of Fourier series, we should expect that $f(x) = F(f)(x)$:

$$f(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x}\, d\xi \tag{6.67}$$

If we try to verify this directly we enter difficulties similar to that in Theorem 6.1: the integral on the right is

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) e^{i\xi(x-t)}\, dt\, d\xi$$

and we cannot apply Fubini's theorem since $\int e^{i\xi(x-t)}\, d\xi$ does not exist. The difficulty is overcome by introducing the convergence factor $e^{-y|\xi|}$, then letting $y \to 0$. More precisely, define the **Poisson transform** of f, a function defined on the upper half plane by

$$Pf(x+iy) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(\xi) \exp[i\xi x - y\,|\xi|]\, d\xi$$

Prove these assertions:

(i) if f is integrable on R,

$$Pf(x+iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{y}{(x-t)^2 + y^2}\, dt$$

and

$$\lim_{y \to 0} Pf(x+iy) = f(x)$$

(*Hint:* Integrate $\int_{-\infty}^{\infty} \exp[i\xi(x-t) - y\,|\xi|]\, d\xi$ over R^+, R^-, independently. The second statement follows as does the result of Theorem 6.1, since this Poisson kernel $y[(x-t)^2 + y^2]^{-1}$ has the same behavior as that for the disk.)

(ii) Pf is harmonic in the upper half plane and thus solves Dirichlet's problem there with the boundary values f.

(iii) if $f \in S(R)$,

$$\lim_{y \to 0} Pf(x+iy) = F(f)(x)$$

so the inversion formula (6.67) holds on $S(R)$.